Quantum noise and quantum Langevin equations

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The quantum Langevin equation of Ford, Kac, and Mazur is rederived and shown to be equivalent to an adjoint equation. This latter can be handled by means of van Kampen's cumulant expansion to yield derivations of the quasiclassical Langevin equation, stochastic electrodynamics, quantum optical, and quantum Brownian motion master equations (under appropriate conditions). The result of Benguria and Kac—that the quantum Langevin equation yields the Boltzmann distribution over energy levels in thermodynamic equilibrium—is also verified.

1. Introduction

The successful invention of the "Langevin equation" by Langevin as long ago as 1908 [1] and the subsequent development of the physics and mathematics of the Langevin equation since then [2–3] into a very useful representation of a variety of noise-related problems has led many authors to propose quantum-mechanical versions [4]. There is no general consensus, however, on what constitutes the "quantum Langevin equation"—as always in quantum mechanics, the subtlety of operator equations leads to a number of interpretations. The classical Langevin equation coexists with the Fokker–Planck equation: There is an exact equivalence between these two formulations of classical noise theory, but in quantum noise theory there is at present no such duality. True, quantum Langevin equations do exist, but are solved by direct methods, and solutions are only possible in very simple situations. The quantum equivalent of the Fokker–Planck equation is the quantum-mechanical master equation, but it is not normally derived from a quantum Langevin equation, and the precise equivalence—though there clearly is some equivalence—has remained obscure.

In this paper I intend to show how such an equivalence may be set up, in at least the most commonly studied kind of quantum noise system: a system with a few degrees of freedom, coupled to a bath of harmonic oscillators.

2. Derivation of the quantum Langevin equation

The model consists of some unspecified system having a small number of degrees of freedom and coupled to a "heat bath" consisting of a large number of harmonic oscillators with a wide range of frequencies. The bath operators are called $p(\omega)$ and $q(\omega)$, and the system operators $\boldsymbol{Z}$. The Hamiltonian for the interacting system and bath is

$$H = H_{\text{sys}} + \frac{1}{2}\int_0^\infty d\omega\{[p(\omega) - \kappa(\omega)X]^2 + \omega^2 q(\omega)^2\}. \tag{1}$$

Here X is a particular system operator and $\kappa(\omega)$ is a smooth function of ω. For simplicity a continuum of frequencies has been assumed from the beginning. It should be noted that by introducing Fourier-transform variables

$$A(t, x) = \sqrt{2/\pi} \int_0^\infty d\omega q(\omega, t)\cos(\omega x/c), \quad \kappa(x) = \sqrt{2/\pi} \int_0^\infty d\omega \kappa(\omega)\cos(\omega x/c), \tag{2}$$

one may write a Lagrangian for (1),

$$L = L_{sys}(Z) + \frac{1}{2} \int_0^\infty dx\{\dot{A}(t, x)^2 - c^2[\partial_x A(t, x)]^2\} + X \int_0^\infty dx\kappa(x)\dot{A}(t, x), \tag{3}$$

which is the Lagrangian for a point system interacting with a one-sided one-dimensional electromagnetic field (essentially a transmission line) by means of a coupling function $\kappa(x)$, which will be significant only near $x = 0$ if $\kappa(\omega)$ is sufficiently smooth. The technique for turning the equations of motion arising from (1) into quantum Langevin equations is almost standard [5]—solve for the field $A(t, x)$ in terms of initial conditions at time t_0 in the remote past, and substitute into the equations of motion for the system operators. The resulting equation of motion, in the limit that $\kappa(\omega)$ is actually a constant, i.e.,

$$\kappa(\omega) = 2f/\pi, \qquad \kappa(x) = 2\sqrt{fc}\,\delta(x), \tag{4}$$

is

$$\dot{Y} = \frac{i}{\hbar} [H_{sys}, Y] + \frac{i}{2\hbar} [X, [f\dot{X} - 2\sqrt{fc}\dot{A}_{in}(t), Y]_+] \tag{5}$$

$$= \frac{i}{\hbar} [H_{sys}, Y] + \frac{i}{2\hbar} [f\dot{X} - 2\sqrt{fc}\dot{A}_{in}(t), [X, Y]]_+, \tag{6}$$

and here Y is an arbitrary system operator, while

$$A_{in}(t) = (2\pi c)^{-\frac{1}{2}} \int_0^\infty d\omega\{q(\omega, t_0)\cos\omega(t - t_0) + \omega^{-1}p(\omega, t_0)\sin\omega(t - t_0)\}, \tag{7}$$

$$\dot{X} = \frac{i}{\hbar} [H_{sys}, Y], \tag{8}$$

and the last equation is obvious from (6). At first glance (5) and (6) appear to be inequivalent, but in the derivation it becomes clear that they are equivalent if the system and the bath represent independent degrees of freedom; i.e., if for all $Y(t)$

$$[Y(t), q(\omega, t)] = [Y(t), p(\omega, t)] = 0. \tag{9}$$

To obtain the more usual form of quantum Langevin equation, consider the case where

$$H_{sys} = \frac{p^2}{2m} + V(x), \quad X = x, \tag{10}$$

where p and x obey canonical commutation relations. One soon derives

$$\dot{x} = p/m, \quad \dot{p} = -V(x) - fp/m + E(t), \tag{11}$$

where

$$E(t) = 2\sqrt{fc}\dot{A}_{in}(t). \tag{12}$$

One also has the commutation relation

$$[\dot{A}_{in}(t), \dot{A}_{in}(t')] = \frac{i\hbar}{2c}\frac{d}{dt}\delta(t - t'), \tag{13}$$

and, if the bath density matrix is thermal with temperature T at time t_0, then

$$\langle[E(t), E(t')]_+\rangle = \frac{\hbar f}{\pi} \int_0^\infty d\omega\omega\coth(\hbar\omega/2kT)\cos\omega(t - t'), \tag{14}$$

where the average is over the bath density matrix. The quantum-mechanical Langevin equation (11), with $E(t)$ specified by (12)–(14), represents a generally accepted concept of what a quantum-mechanical Langevin equation should be. This equation has quite a lengthy history. Senitzky [6] was apparently the first to propose it, and it was rather rigorously derived from the same kind of basis as I have presented by Ullersma [7] and by Ford, Kac, and Mazur [8]. The most recent significant contribution to its development was by Benguria and Kac [9]. These authors were able to show that the quantum Gaussian nature of $E(t)$ and the relations (13) and (14) would guarantee that, in the low-friction limit, the system density matrix approached the appropriate Boltzmann distribution over energy levels for a restricted class of potentials chosen for their mathematical tractability. Their method and result are correct, but perhaps a little remote from the way such problems are usually tackled in practice.

The most thorough bibliographical analysis of this subject has been undertaken by Dekker [10] in the more limited context of the damped harmonic oscillator. Caldeira and Leggett [11] have also summarized the history, and make the comment that "the general question of the validity of the 'quantum Langevin equation' outside the especial case of the harmonic oscillator is a very open one."

The aim of this paper is to settle the "open questions" of the validity of the quantum Langevin equation by showing that it leads directly to a large number of well-attested results, in particular in the field of quantum optics but in other fields as well.

3. Preservation of commutation relations

One of the perennial questions which arise in quantum damping and noise equations is the question of the preservation of commutation relations. Quantum mechanics only allows a unitary evolution of Heisenberg operators, so the form of the equal-time commutation relations of the operators $Y(t)$ must be invariant in time. The most obvious example of this fact is that of the canonical commutation relations $[x(t), p(t)] = i\hbar$, but the requirement is in fact far more general. It was very early realized that a damping equation without noise would lead to the decay of the canonical commutation relations, and the solution to this phenomenon was proved by the choice of an operator noise with the correct commutation relations. [The choice (13), now considered correct, was not arrived at without some controversy.] Nevertheless, to my knowledge the only explicit proofs of the result are for the very simple case of the harmonic oscillator, and so far no general proof has been given. In this section I shall prove the result in general.

An intermediate step in the derivation of the quantum Langevin equation (5), (6) is the equation of motion

$$\dot{Y} = \frac{i}{\hbar}[H_{\text{sys}}, Y] + \frac{i}{2\hbar}\int_0^\infty d\omega[[Y, p(\omega) - \kappa(\omega)X]_+, \kappa(\omega)X] \tag{15}$$

which can be written

$$\dot{Y} = \frac{i}{\hbar}\left[H_{\text{sys}} - \int_0^\infty d\omega\left(\kappa(\omega)p(\omega)X - \frac{1}{2}\kappa(\omega)X^2\right), Y\right] \tag{16}$$

provided $p(\omega, t)$ commutes with all $Y(t)$. This is, of course, true, because $p(\omega, t)$ and $Y(t)$ represent canonical operators for different degrees of freedom, and their equal-time commutator must vanish. Thus the equation of motion for all operators Y is unitary, and hence all commutation relations between different Y are preserved.

However, as a proof this begs the question. The original Hamiltonian (1) obviously defines a unitary evolution, and thus, by construction, commutation relations must be preserved. What we really want to prove is the following: Let us suppose we are given

1. The quantum Langevin equation in the form (6), in which $A_{\text{in}}(t)$ is an operator function satisfying the commutation relation (13).
2. The initial-condition commutation relation

$$[A_{\text{in}}(t), Y(t_0)] = 0. \tag{17}$$

Condition (17) is physically reasonable. It corresponds to a situation in which the system and the noise are uncoupled until time t_0, and therefore the system operators at that time are independent of the noise. [Alternatively, from the point of view of the Hamiltonian (1), the noise is defined in terms of the bath operators at time t_0, which commute with the system operators at the same time.]

Given these two conditions, can it be proven that the $Y(t)$ which arise as solutions of the quantum Langevin equation with initial conditions at t_0 continue to satisfy the condition of preservation of commutation relations? The method of proof depends on reconstructing the $p(\omega, t)$ from the noise and the initial conditions on the $Y(t)$, and hence showing that evolution in the form (16) is unitary. In order to avoid ambiguities it is necessary to use a $\kappa(\omega)$ which is not constant, so that $\kappa(x)$ is not the singular delta function of (4), but some close approximation to it. The proof is valid for $\kappa(x)$ as close to a delta function as we please, however.

Under these circumstances, the quantum Langevin equation takes the form

$$\begin{aligned}\dot{Y} &= \frac{i}{\hbar}[H_{\text{sys}}, Y] - \frac{i}{2\hbar} \\ &\quad \times\left[X, \left[Y, \xi(t) - \int_{t_0}^t f(t-t')\dot{X}(t')dt' - f(t-t_0)X(t_0)\right]_+\right] \\ &= \frac{i}{\hbar}[H_{\text{sys}}, Y] - \frac{i}{2\hbar} \\ &\quad \times\left[[X, Y], \xi(t) - \int_{t_0}^t f(t-t')\dot{X}(t')dt' - f(t-t_0)X(t_0)\right]_+,\end{aligned} \tag{18}$$

in which

$$\xi(t) = i\int_0^\infty d\omega\kappa(\omega)\sqrt{\hbar\omega/2} \times [-a(\omega)(t_0)e^{-i\omega(t-t_0)} + a(\omega)'(t_0)e^{i\omega(t-t_0)}] \tag{19}$$

and

$$f(t) = \int_0^\infty d\omega\kappa(\omega)^2\cos(\omega t), \tag{20}$$

and under these conditions

$$[\xi(t), \xi(t')] = i\hbar\frac{d}{dt}f(t-t'). \tag{21}$$

Now we need the relation

$$\int_0^\infty d\omega\kappa(\omega)p(\omega, t) = \xi(t) - \int_{t_0}^t X(t')\frac{d}{dt}f(t-t')dt', \tag{22}$$

which can be proved by solving the equations of motion for $p(\omega, t)$, $q(\omega, t)$ arising from the Hamiltonian (1). We can derive the quantum Langevin equation by substituting Equation (22) into (16).

Now turn this round the other way. Assume that we have the quantum Langevin equation and $\xi(t)$: Let us *define* $\int_0^\infty d\omega\kappa(\omega)p(\omega, t)$ by (22). If this expression is to commute with $Y(t)$, then from (22) we can see that this is equivalent to the requirement

$$\left[Y(t), \xi(t) - \int_{t_0}^t X(t')\frac{d}{dt}f(t-t')dt'\right] = 0. \tag{23}$$

Let us prove a slightly stronger result. Assume that for all s and all u such that $t \geq u \geq t_0$,

$$\left[Y(u),\, \xi(s) - \int_{t_0}^{u} dt' X(t') \frac{d}{ds} f(s - t') \right] = 0 \tag{24}$$

for all system operators $Y(t)$. Then we want to show that

$$\left[Y(t + dt),\, \xi(s) - \int_{t_0}^{t+dt} dt' X(t') \frac{d}{ds} f(s - t') \right] = 0. \tag{25}$$

That is, if the relation (25) is true in the interval $t \geq u \geq t_0$, then it is also true just outside the upper end of the interval, and consequently is true for all $t \geq t_0$. Now (25) is equivalent to

$$\left[\dot{Y}(t),\, \xi(s) - \int_{t_0}^{t} dt' X(t') \frac{d}{ds} f(s - t') \right]$$
$$= [Y(t), X(t)] \frac{d}{ds} f(s - t). \tag{26}$$

We substitute for $Y(t)$ using (18) and do a partial integration, which brings the left-hand side of (26) into the form

$$\left[\frac{i}{\hbar} [H_{\text{sys}}, Y] - \frac{i}{2\hbar} \left[X, \left[Y, \xi(t) - \int_{t_0}^{t} dt' X(t') \frac{d}{dt} f(t - t') - f(0) X(t) \right] \right]_+ ,\right.$$
$$\left. \xi(s) - \int_{t_0}^{t} dt' X(t') \frac{d}{ds} f(s - t') \right]. \tag{27}$$

Because (25) is true for *all* system operators, we can reduce (27) to

$$-\frac{i}{2\hbar} \left[X, \left[Y, \left[\xi(t) - \int_{t_0}^{t} dt' X(t') \frac{d}{dt} f(t - t'),\right.\right.\right.$$
$$\left.\left.\left. \xi(s) - \int_{t_0}^{t} dt' X(t') \frac{d}{ds} f(s - t') \right] \right]_+ \right]. \tag{28}$$

The innermost commutator is very simple—using (24), it expands to

$$[\xi(t), \xi(s)]$$
$$- \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' f'(t - t') f'(s - t'') [X(t'), X(t'')]$$
$$- \int_{t_0}^{t} dt'' \int_{t_0}^{t''} dt' f'(t - t') f'(s - t'') [X(t'), X(t'')]$$
$$+ \int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' f'(t - t') f'(s - t'') [X(t'), X(t'')]. \tag{29}$$

The integrands are all the same, and the domains of integration of the first two integrals add up to that of the third, so the last three terms all cancel out, leaving only the first, which is a nonoperator quantity given by (21). Simplifying, we find that (28) is equal to the right-hand side of (26), which is what we wanted to prove.

Thus, assuming (24) for $t \geq u \geq t_0$ implies the truth of (24) for all t. Setting $u = t_0$, we see that (24) becomes in that case simply $[\xi(s), Y(t_0)] = 0$, which is true by hypothesis.

In summary, the quantum Langevin equation and $[\xi(s), Y(t_0)] = 0$ lead to the conclusion that $\int_0^\infty d\omega \kappa(\omega) p(\omega, t)$ always commutes with all system operators, and hence that the quantum Langevin is expressible in terms of a commutator, which represents a unitary evolution and thus preserves the commutation relations.

4. The adjoint equation

In classical physics, the derivation of a Fokker–Planck equation equivalent to the classical version of the Langevin equations (11) can be done directly via stochastic differential equation theory, using the fact that classically $E(t)$ is delta-correlated:

$$\langle E(t) E(t') \rangle = 2 f k T \pi \delta(t - t'). \tag{30}$$

However, we know that physically the noise is not delta-correlated but merely has a short correlation time. In this case, the equation for a distribution function $P(x, p, t)$ as a function of time is Kubo's stochastic Liouville equation [3, 11, 12]

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x} \left(\frac{p}{m} P \right) + \frac{\partial}{\partial p} ([V'(x) + fp/m - E(t)] P), \tag{31}$$

which is exactly equivalent to the equations (11) provided the fluctuating $E(t)$ is not too singular. In the limit that $E(t)$ approaches white noise, one can show that the average of $P(x, p, t)$ over $E(t)$ obeys a Fokker–Planck equation. In order to carry out this kind of program quantum-mechanically we need an analog of the stochastic Liouville equation. The derivation of this analogous equation does not come from the equations in the form (11), but rather from the equations in the form (6), which are *linear* equations for an arbitrary operator Y. [In fact, the same is true classically—the stochastic Liouville equation is derivable from the equation for an arbitrary function $Y(x, p)$ of x and p rather than from the equations for x and p themselves.] We assume that the bath and system are initially independent; hence the density operator in the Heisenberg picture is $\rho_{\text{sys}} \otimes \rho_{\text{B}}$. Now let $Y(t)$ be an arbitrary system operator in the Heisenberg picture, and Y a Schrödinger-picture representation of the same operator. Then we can consistently define a quantity $\mu(t)$ by

$$\text{Tr}_s\{ Y(t) \rho_{\text{sys}} \otimes \rho_{\text{B}} \} = \text{Tr}_s\{ Y \mu(t) \} \rho_{\text{B}}, \tag{32}$$

if the quality is true for all operators (i.e., a complete set) Y and $Y(t)$. From (32), we get

$$\left.\begin{aligned} \text{Tr}_s\{ E(t) Y(t) \rho_{\text{sys}} \otimes \rho_{\text{B}} \} &= E(t) \text{Tr}_s\{ Y \mu(t) \} \rho_{\text{B}}, \\ \text{Tr}_s\{ Y(t) E(t) \rho_{\text{sys}} \otimes \rho_{\text{B}} \} &= \text{Tr}_s\{ Y \mu(t) \} E(t) \rho_{\text{B}}, \end{aligned}\right\} \tag{33}$$

and noting that

$$\text{Tr}_s\{ \dot{Y}(t) \rho_{\text{sys}} \otimes \rho_{\text{B}} \} = \text{Tr}_s\{ Y \dot{\mu}(t) \} \rho_{\text{B}}, \tag{34}$$

we derive from (5) the *adjoint equation*

$$\dot{\mu}(t) = [H_{\text{sys}}, \mu(t)] + \frac{i}{2\hbar} [[f \dot{X} - E(t), \mu(t)]_+, X], \tag{35}$$

which is what we want. Clearly, the Schrödinger-picture-system density matrix is given by

$$\rho_S(t) = \mathrm{Tr}_B\{\mu(t)\rho_B\} \equiv \langle\mu(t)\rangle. \tag{36}$$

Thus, $\mu(t)$ is a kind of quantum stochastic density matrix in which the quantity $\mu(t)$ is a function of the operator quantity $E(t)$, the impressed quantum noise.

5. A commuting representation of quantum noise

Rather miraculously, the operator nature of $E(t)$ can be almost completely eliminated. From (35) we can see that $E(t)$ only arises as an anticommutator. If we define an operator $\alpha(t)$ by

$$\alpha(t)\mu(t) \equiv \tfrac{1}{2}\,[E(t),\, \mu(t)]_+, \tag{37}$$

then, in fact,

$$\alpha(t)\alpha(t') = \alpha(t')\alpha(t), \tag{38}$$

a fact that arises from the c-number nature of the commutator (27). The multiplication $\alpha(t)\alpha(t')$ is defined by (37) as associative, so that $\alpha(t)$ is in fact equivalent to a c-number random function. The actual statistics of $\alpha(t)$ depend on the density matrix of the bath: In the case where this is thermal, $\langle\alpha(t)\alpha(t')\rangle$ is given by the right-hand side of (14).

This means that we can write the adjoint equation in the form

$$\dot\mu(t) = A_0\mu(t) + A_1\alpha(t)\mu(t), \tag{39}$$

where A_0 and A_1 are linear operators in the system space and are defined by

$$\left.\begin{aligned} A_0\mu(t) &= \frac{i}{\hbar}\,[H_{\mathrm{sys}},\, \mu(t)] + \frac{i}{2\hbar}\,[[f\dot X,\, \mu(t)_+],\, X],\\ A_1\mu(t) &= \frac{i}{\hbar}\,[X,\, \mu(t)]. \end{aligned}\right\} \tag{40}$$

We can now look at the adjoint equation from a number of points of view. It can be used directly in some simple situations, for example, a two-level atom interacting with a one-dimensional electromagnetic field. It is even possible to use ordinary stochastic methods to simulate this adjoint equation numerically. By doing this, it is possible to analyze the influence of an incoming squeezed-light field coming from a degenerate parametric amplifier. The detailed results will be published elsewhere.[1]

It is also possible to employ a number of approximation methods, the subject of the remainder of this paper.

6. The Wigner function and the quasiclassical Langevin equation

The Wigner function is one of many *quasiprobabilities* which enable one to represent quantum mechanics in a form rather closer to classical physics. One can define a c-number distribution function over all canonical variables such that the quantum-mechanical averages of symmetrically ordered products of quantum operators are given by the classical averages over the corresponding variables with respect to this c-number distribution function, known as the Wigner function. It can be shown that the Wigner function need not, however, be positive, and that not all normalizable functions are admissible as Wigner functions.

Suppose the Wigner function corresponding to $\mu(t)$ is $W(x, p, t)$. Then, for example, the symmetrized average of xp is given by

$$\mathrm{Tr}_{\mathrm{sys}}\{\tfrac{1}{2}\,(xp + px)\mu(t)\} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp\,xp\,W(x, p, t). \tag{41}$$

The equation of motion for the Wigner function, in the case where H_{sys} is (12), is derived by the substitution

$$\left.\begin{aligned} p\mu(t) &\to \left(p - \frac{i\hbar}{2}\frac{\partial}{\partial x}\right)W(x, p, t),\\ \mu(t)p &\to \left(p + \frac{i\hbar}{2}\frac{\partial}{\partial x}\right)W(x, p, t),\\ x\mu(t) &\to \left(x + \frac{i\hbar}{2}\frac{\partial}{\partial p}\right)W(x, p, t),\\ \mu(t)x &\to \left(x - \frac{i\hbar}{2}\frac{\partial}{\partial p}\right)W(x, p, t) \end{aligned}\right\} \tag{42}$$

into the adjoint equation. The result is

$$\begin{aligned} \frac{\partial W}{\partial t} &= \left\{-\frac{\partial}{\partial x}\frac{p}{m} + \frac{\partial}{\partial p}\,[V'(x) + fp/m - \alpha(t)]\right\}W\\ &\quad + \left\{\sum_{n=1}^{\infty}\left(\frac{i\hbar}{2}\right)^{2n}\frac{\partial^{2n+1}}{\partial x^{2n+1}}\,V^{2n+1}(x)\right\}W. \end{aligned} \tag{43}$$

There are a number of situations in which the second line vanishes or is negligible:

1. If $V(x) = Ax + Bx^2$; i.e., we are dealing with a harmonic oscillator, a linear potential, or a free particle.
2. If $h \to 0$.
3. If $f \to \infty$: the large-friction case. In this case $\alpha(t)$ also becomes larger, and this represents the dominant noise term. The last line in (43) is a kind of noise, which is independent of f and is then presumably negligible in this limit. (This argument certainly lacks rigor, but can probably be made rigorous.)

In all these cases, we are left with a conventional stochastic Liouville equation for the Wigner function, equivalent to the classical Langevin equation

$$\left.\begin{aligned} \dot x &= p/m,\\ \dot p &= -V'(x) - fp/m + \alpha(t), \end{aligned}\right\} \tag{44}$$

[1] A. S. Parkins and C. W. Gardiner, "Finite Bandwidth Effects in the Inhibition of Atomic Phase Delays by Squeezed Light," to be published.

in which, however, the noise spectrum is given by the right-hand side of (14). This equation has been called the quasiclassical Langevin equation by Schmid [13]. It has been used by Koch, van Harlingen, and Clarke [14] to analyze (significantly, in the high-friction limit) their experiments on quantum noise, and has been used extensively in superconductivity theory. It is therefore respectable, but its validity is limited. In fact, there does exist a body of knowledge called "stochastic electrodynamics," in which various authors [15] have built up a theory in which a classical particle interacts with a random EM field whose statistics are chosen to give a Planck spectrum—and this describes precisely the nature of (44). Exact agreement between stochastic electrodynamics and quantum theory is found for assemblies of harmonic oscillators and for free particles. From my point of view this is not surprising—but the terms in the second line of (43) will make their presence felt in all other cases, and stochastic electrodynamics cannot be a valid representation of reality for general situations. (In fact, a three-dimensional version of this theory gives SED exactly in the harmonic limit.)

7. Master equations

The adjoint equation in the form (39), (40) immediately brings to mind van Kampen's cumulant expansion for linear stochastic differential equations [3, 12]. In the limit of short correlation time, an equation can be derived for $\rho_S(t) = \langle \mu(t) \rangle$, in the form

$$\dot{\rho}_S(t) = A_0 \rho_S(t) + \int_0^\infty d\tau A_1 e^{A_0 \tau} A_1 e^{-A_0 \tau} \langle \alpha(t) \alpha(t-\tau) \rangle \rho_S(t), \tag{45}$$

which is valid in the case where $\| A_1 \| \; \| \alpha \| \tau_c$ is small. Here τ_c is the correlation time of $\alpha(t)$, and $\| A_1 \|$ and $\| \alpha \|$ are measures of the "size" of these two operators. The relevant correlation time is the *thermal correlation time* τ_T, which is given by the asymptotic form of the correlation function (14) of $E(t)$, that is,

$$\langle E(t) E(t') \rangle \sim -\frac{4 f \pi^2 k^2 T^2}{\hbar} \exp\left(-\frac{2\pi k T \, |t - t'|}{\hbar} \right), \tag{46}$$

giving a thermal correlation time

$$\tau_T = \hbar / 2\pi k T \tag{47}$$

as first noted by Ullersma [7]. The value of $\| \alpha \|$ is given by the square root of the coefficient of the exponential in the asymptotic form, i.e.,

$$\| \alpha \| = 2\pi k T \sqrt{f/\hbar}. \tag{48}$$

To estimate the size of the operator A_1 is then all that remains. If we are dealing with a particle in a potential well, as in Equation (10), then the operator A_1 is given by

$$A_1 = \frac{i}{2\hbar} [x, \;] \rightarrow -\frac{1}{2} \frac{\partial}{\partial p}. \tag{49}$$

If we are near thermal equilibrium, the p dependence of $\mu(t)$ is $\exp(-p^2/2mkT)$, so we can estimate

$$\left\| \frac{1}{2} \frac{\partial}{\partial p} \right\| \approx \frac{1}{2\sqrt{mkT}}. \tag{50}$$

Putting these together gives a condition for the validity of (45),

$$\hbar f / m \ll kT, \tag{51}$$

which is equivalent to

$$\tau_T \ll \tau_D, \tag{52}$$

where

τ_D is the damping constant of the system $= m/f$.

$$\tau_T \text{ is the thermal correlation time of } E(t). \tag{53}$$

These are admittedly rather crude estimates, but should suffice for our purposes. The condition (52) is in the end rather reasonable—it simply requires that the correlation time of the noise be much less than the typical time scale of the damped motion of the system. Two cases can now be distinguished.

8. The quantum optical case

We resolve operators in (9) into eigenoperators of H_{sys}, namely

$$X = \sum_m (X_m^+ + X_m^-), \tag{54}$$

where

$$[H_{sys}, X_m^\pm] = \pm \hbar \omega_m X_m^\pm. \tag{55}$$

(This is always possible if the eigenvectors of H_{sys} form a complete set.) If we now consider the case when all the ω_m are much larger than f (as is always the case in quantum optics), then we can omit the f-dependent term in A_0 [as defined in (40)] in the terms $e^{\pm A_0 \tau}$ in (45), and thus make the replacement

$$e^{\pm A_0 \tau} X_m^\pm \rightarrow e^{\pm i \omega_m \tau} X_m^\pm e^{\pm A_0 \tau}. \tag{56}$$

Using the correlation function (14), one finds, after some labor, the conventional quantum optical master equation

$$\begin{aligned} \dot{\rho}_S(t) = & -\frac{i}{\hbar} [H_{sys}, \rho_S] \\ & - \sum_m \frac{\pi \omega_m}{2\hbar} (\bar{N}(\omega_m) + 1) \kappa(\omega_m)^2 [\rho_S X_m^+ - X_m^- \rho_S, X] \\ & - \sum_m \frac{\pi \omega_m}{2\hbar} \bar{N}(\omega_m) \kappa(\omega_m)^2 [\rho_S X_m^- - X_m^+ \rho_S, X] \\ & + \frac{i}{2\hbar} \sum_m P \int_{-\infty}^{\infty} \frac{d\omega \kappa(\omega)^2}{\omega_m - \omega} \left(\bar{N}(\omega) + \frac{1}{2} \right) [[\rho_S, X_m^+ - X_m^-], X] \\ & + \frac{i}{4\hbar} \sum_m P \int_{-\infty}^{\infty} \frac{d\omega \kappa(\omega)^2}{\omega_m - \omega} \left(\bar{N}(\omega) + \frac{1}{2} \right) [[\rho_S, X_m^+ + X_m^-]_+, X], \end{aligned} \tag{57}$$

where $\bar{N}(\omega) = [\exp(\omega/kT) - 1]^{-1}$. [Strictly, we can only derive from the quantum Langevin equation (6) an equation like (57) in which $\kappa(\omega) = 2f/\pi$. However, similar analysis in the case where $\kappa(\omega)$ is frequency-dependent leads to a friction term of the kind $\int_{-\infty}^{t} f(t - t')X(t')dt'$, and a corresponding noise term which in turn leads to (57)]. Only the second two terms in (57) represent damping. The first term is of course the systematic motion, while the two final terms are a combination of Lamb shift and Stark shift terms.

This kind of master equation is capable of describing almost all phenomena which can be experienced in quantum optics, although it is not often written down in precisely the form I have given. As used in quantum optics, it is basically due to Louisell [16]. Usually the rotating wave approximation is also made; it involves the following:

1. Define an interaction picture system density operator by

$$\rho_I(t) = \exp\left(\frac{i}{\hbar} H_{sys}t\right)\rho_S(t)\exp\left(-\frac{i}{\hbar} H_{sys}t\right). \tag{58}$$

   Note that the interaction picture master equation no longer has a term corresponding to the first line in (57), and use the relation (55) to commute $\exp(\pm i/\hbar H_{sys}t)$ with $X_m^{\pm}$. This leaves certain terms with factors like $\exp[i(\omega_m - \omega_n)t]$—a very rapid time variation on the time scale of atomic decays, which allows us to neglect them completely.
2. The Lamb and Stark shift terms are dropped, since they are very small, leaving the interaction picture master equation in the form

$$\begin{aligned}\dot{\rho}_I = \sum_m \frac{\pi\omega_m}{2\hbar} [\bar{N}(\omega_m) + 1]\kappa(\omega_m)^2(2X_m^-\rho_I X_m^+ - \rho_I X_m^+ X_m^- \\ - X_m^+ X_m^- \rho_I) \\ + \sum_m \frac{\pi\omega_m}{2\hbar} \bar{N}(\omega_m)\kappa(\omega_m)^2(2X_m^+\rho_I X_m^- - \rho_I X_m^- X_m^+ \\ - X_m^- X_m^+ \rho_I).\end{aligned} \tag{59}$$

   In this form, the master equation describes transitions in an atomic system in a radiation field. Adaptations to include small additional nonlinearities and driving fields are commonly made by adding terms as follows.

- *Driving fields—Inputs and outputs*

It is very common to consider a situation where a laser beam is incident on an atom. This means that the heat bath (in this case the electromagnetic field) is no longer characterized as having thermal statistics, but has as well some coherent excitation in a small range of modes. It is helpful to go back to the transmission line model of Section 2, given by the Lagrangian (3), and to view this as a one-dimensional model of electrodynamics. After some standard wave equation theory, it is possible to show that the field $A(t, x)$ is given (in the limit that $t_0 \to \infty$) by

$$\begin{aligned}A(t, x) = A_{in}(t + x/c) + A_{in}(t - x/c) \\ - \frac{1}{2}\int_{-\infty}^{\infty} d\tau\kappa(\tau)X(t - |\tau - x/c|),\end{aligned} \tag{60}$$

and it is clear that the three terms correspond to incoming, reflected, and radiated fields. By solving the equations in terms of a *final* condition t_1 in the remote future, one can similarly construct an "out" field in terms of which

$$\begin{aligned}A(t, x) = A_{out}(t + x/c) + A_{out}(t - x/c) \\ + \frac{1}{2}\int_{-\infty}^{\infty} d\tau\kappa(\tau)X(t + |\tau - x/c|),\end{aligned} \tag{61}$$

and it is not difficult to see that in the region where $\kappa(x)$ is zero, i.e., away from the region of interaction between field and system,

$$A(t, x) = A_{in}(t + x/c) + A_{out}(t - x/c) \tag{62}$$

and

$$A_{in}(t) = A_{out}(t) + \frac{1}{2}\int_{-\infty}^{\infty} d\tau\kappa(c\tau)X(t - \tau). \tag{63}$$

From all this we see that the damping-noise interpretation also has an "input-output" interpretation. The quantum Langevin equations (5) and (6) can also be interpreted as corresponding to a system driven by the incoming "in" field, and losing energy via radiation damping into the "out" field.

From this point of view, the inclusion of a coherent driving field is no problem, since we need only specify the "in" field. One simply makes the requirement

$$\langle A_{in}(t)\rangle = a_{in}(t), \tag{64}$$

which gives the mean time-dependent excitation. The statistics of any fluctuations can be specified by setting

$$E(t) = 2\sqrt{fc}\{\dot{A}_{in}(t) - \dot{a}_{in}(t)\}, \tag{65}$$

and specifying the relevant correlation functions of $E(t)$. For example, a coherent driving field superimposed on a thermal background is obtained by taking the correlation function (14) for $E(t)$ as defined by (65). The quantum-mechanical Langevin equation corresponding to (6) becomes

$$\begin{aligned}\dot{Y} = \frac{i}{\hbar}[H_{sys}, Y] - \frac{2i\sqrt{fc}}{\hbar}\dot{a}_{in}(t)[X, Y] \\ + \frac{i}{2\hbar}[f\dot{X} - E(t), [X, Y]]_+,\end{aligned} \tag{66}$$

and the master equations (57), (59) acquire an extra term,

$$\frac{2i\sqrt{fc}}{\hbar} a_{in}(t)[X, \rho]. \tag{67}$$

This corresponds to simply adding to the system Hamiltonian a corresponding driving term.

- *Small anharmonicity*

It is common in quantum optics to consider the system as consisting of a single mode of the electromagnetic field inside a cavity, which communicates with the electromagnetic bath and driving modes through an almost-perfect mirror. The system would then be a perfect harmonic oscillator; however, one also introduces some kind of weak nonlinearity via a nonlinear medium within the cavity. It is thus possible to write

$$H_{sys} = H_0 + H_{nl}, \tag{68}$$

where H_{nl} is very small compared to H_{sys}. How does this affect the analysis? Typically the effects of H_{nl} are of the same order of magnitude as the damping, so H_{nl} can be neglected in all the procedures leading to (57), (59). In particular this means that

1. X_m are eigenoperators of H_0.
2. ω_m are the transition frequencies of H_0.

This means that the relevant interaction picture is defined in terms of H_0, so that (57) is modified simply by adding a term

$$-\frac{i}{\hbar}[H_{nl}, \rho_1]. \tag{69}$$

Notice that there is an interesting transition region between small nonlinearity, which has this effect, and large nonlinearity, which modifies the whole master equation by modifying the relevant energy levels.

- *Stationary solution—Boltzmann distribution*

If we neglect the Stark and Lamb shift terms in the master equation (57), the stationary solution is obviously the Boltzmann distribution, for the equations (57) necessarily imply that

$$\begin{aligned}\exp(-H_{sys}/kT)X_m^{\pm} \\ = \exp(\pm\hbar\omega_m/kT)X_m^{\pm}\exp(-H_{sys}/kT),\end{aligned} \tag{70}$$

from which, using the definition of $\bar{N}$ in (57), it is obvious that the corresponding terms in the two summations cancel each other. A general and correct inclusion of the effects of the Lamb and Stark shifts is more tricky, but cannot alter this conclusion in the lowest order. We comment now on the result of Benguria and Kac [9] that the Boltzmann distribution solution for the stationary state requires quantum Gaussian statistics for $E(t)$ in the case (as here) where $[E(t_1), E(t_2)]$ is a c-number. My admittedly much less rigorous but certainly far more physically transparent result does not seem to require Gaussian statistics, since only the correlation functions are involved. This is much the same as in the classical case, for there the proof that there is a white-noise limit of a non-white-noise stochastic Liouville equation that does not require a Gaussian physical noise. Nevertheless, the resultant Fokker–Planck equation is exactly equivalent to a white-noise stochastic differential equation with Gaussian noise.

9. The quantum Brownian motion case

It may be that A_0 is very small; that is, $\exp(A_0t)$ does not differ significantly from 1 over a correlation time of $\alpha(t)$. In that case we can drop the exponential terms in (57), and there is no longer any need to introduce $X_m^{\pm}$ operators. The master equation becomes

$$\begin{aligned}\dot{\rho}_S = -\frac{i}{\hbar}[H_{sys}, \rho_S] + \frac{i}{2\hbar}[[f\dot{X}, \rho_S]_+, X] \\ + \frac{fkT}{\hbar^2}[[X, \rho_S], X]\end{aligned} \tag{71}$$

which is a form that has been proposed by many authors [17]. It can also be viewed as a high-temperature limit of the quantum optical master equation.

The assumption that leads to (71) requires that all eigenfrequencies ω_m be much less than the correlation time of $E(t)$, which is easily shown to be equivalent to

$$\hbar\omega_m \ll kT. \tag{72}$$

Notice that this condition and the condition (51) for the validity of the method are independent. This means that both weak and strong damping can be treated by this equation, unlike the quantum optical master equation, which requires weak damping.

- *The small $\hbar z$ case*

If we consider the function

$$P(u, z) = \langle u + \tfrac{1}{2}\hbar z|\rho|u - \tfrac{1}{2}\hbar z\rangle \tag{73}$$

we find that the master equation can be written as a quite simple partial differential equation,

$$\begin{aligned}\frac{\partial P}{\partial t} = \Bigg\{\frac{i}{m}\frac{\partial^2}{\partial u\partial z} - \frac{i}{\hbar}\left(V\left(u + \frac{1}{2}\hbar z\right)\right. \\ \left.- V\left(u - \frac{1}{2}\hbar z\right)\right) - \frac{fz}{m}\frac{\partial}{\partial z} - fkTz^2\Bigg\} P.\end{aligned} \tag{74}$$

If $\hbar z$ is considered small, we can approximate

$$V(u + \tfrac{1}{2}\hbar z) - V(u - \tfrac{1}{2}\hbar z) \approx \hbar z V'(u). \tag{75}$$

In this case, one can compute the stationary solution to be

$$P(u, z) = \exp\left\{-\frac{V(u)}{kT} - mkTz^2\right\}. \tag{76}$$

Notice that this gives an almost diagonal intensity matrix at any finite temperature, since the exponential fall-off away from the diagonal takes place on a distance scale of the order of $\hbar\sqrt{2mkT}$. It is interesting to note that this is quite different from the usual dependence on Planck's constant which arises from the WKB approximation applied to a system without noise and damping.

Of course this result is really a purely classical result, in the sense that the corresponding Wigner function is just the classical canonical distribution and does not involve Planck's constant. We must go to higher orders to see genuine quantum effects.

• *Higher-order corrections*

By taking (75) to the next order in Planck's constant, we arrive at the equation

$$\frac{\partial P}{\partial t} = \left\{\frac{i}{m}\frac{\partial^2}{\partial u \partial z} - iV'(u)z - i\frac{\hbar^2}{24}V'''(u)z^3 - \frac{fz}{m}\frac{\partial}{\partial z} + fkTz^2\right\}P. \tag{77}$$

This equation can be solved by approximation methods. As an example, consider the high friction limit, $f \to \infty$. One can use the standard adiabatic elimination techniques [3] to find an equation for

$$\bar{P} \equiv P(u, 0) = \langle u|\rho|u\rangle, \tag{78}$$

that is, for the probability distribution. The equation derived by this method is

$$\frac{\partial \bar{P}}{\partial t} = \frac{m}{f}\left\{\frac{\partial}{\partial u}[1 + mf^{-2}V''(u)]\left[\frac{V'(u)}{m} + \frac{kT}{m}\frac{\partial}{\partial u}\right]\bar{P} - \frac{\hbar^2}{24}\frac{m}{8f^3}\frac{\partial^3}{\partial u^3}(V'''(u)\bar{P})\right\}. \tag{79}$$

This equation is a quantum-corrected version of the corrected Smoluchowski equation [3]. In the case of a sinusoidal potential

$$V(u) = V_0\cos(\alpha u), \tag{80}$$

the stationary solution is approximately given by

$$\bar{P}_s \approx \exp(-U(u)/kT), \tag{81}$$

with

$$U(u) = V_0\cos(\alpha u)\left[1 - \frac{\epsilon a^4}{m^2}\left\{\frac{4}{3}\left(\frac{V_0}{kT}\right)^2\cos^2(\alpha u) - 3\left(\frac{V_0}{kT}\right)\cos(\alpha u) - \left[4\left(\frac{V_0}{kT}\right)^2 - 1\right)\right\}\right] \tag{82}$$

and

$$\epsilon = \hbar^2/24. \tag{83}$$

When a Josephson-junction model is used, the parameters are

$$\left.\begin{array}{ll} V_0 = I_0/2e, & \alpha = 2e/\hbar C, \\ T = 10^{-1}\text{ deg K}, & I_0 = 10\ \mu\text{A}, \\ f/m = (RC)^{-1}, & M = C, \\ R = 200\ \Omega, & C = 4.7 \times 10^{-11}\text{ F}, \end{array}\right\} \tag{84}$$

which gives

$$\epsilon \approx 5 \times 10^{-7}, \qquad V_0 \approx 1.5 \times 10^4. \tag{85}$$

Substituted into the result for the effective potential (82), this gives a noticeable correction to the simple Boltzmann result, which might perhaps be observable. The full analysis of the predictions of this master equation are still in progress, and will be published elsewhere.[2]

• *Damping of quantum coherence*

Savage and Walls [18] have used the quantum Brownian master equation to study the free particle and the harmonic oscillator, and have shown that the effect of damping as introduced by this equation very rapidly reduces the density matrix corresponding to a macroscopic superposition of quantum states to a diagonal density matrix corresponding to a mixed state. No precise comparison has been made between their results and experiment. It should be noted that their comparison between electron diffraction and the predictions of their calculations is flawed by a numerical error—in fact, redoing their arithmetic, I find that the observed diffraction patterns, which are not absolutely sharp, could well be predicted by quite reasonable values of the parameters.

10. Conclusions

What has been achieved here? First, we have a full link-up between the master equation and the quantum Langevin methods in quantum noise theory. Of particular utility is the new form of the adjoint equation (39). This provides, via the acknowledgement that $\alpha(t)$ is essentially a c-number quantity, a link with the methods of classical stochastics. A second achievement is the recognition that stochastic electrodynamics arises from the truncation (43). Thus the successes of SED are bound up in the special nature of the problems tackled. What would SED do wrongly? This becomes clear when we realize that not including the last terms amounts to treating the system classically. By following van Kampen's methods with this truncated form we would not find the frequencies ω_m turning up (the transition frequencies), but would find rather that the relevant frequencies were the classical frequencies. Only for the harmonic oscillator do classical and quantum frequencies coincide.

A third result is the elucidation of the quasiclassical Langevin equation as having the same status as SED, except that it may be valid in a high friction limit. Finally, we emphasize that the methods are more than merely elegant formalism. By using the adjoint equation, one can compute correlations between noise and system variables—a result of some importance if the noise source is a squeezed light beam

[2] C. W. Gardiner and M. L. Steyn-Ross, "Quantum Corrections to the Kramers Problem," to be published.

[19]. Simulations are also possible. All the methods are valid for such nonthermal heat baths—only the details of the particular bath correlation functions need changing. Finally, to my knowledge, this is the first time that the possibility of seriously analyzing the quantum Brownian motion master equation for situations which are neither harmonic nor free has been proposed.

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