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On the Velocity of a Domain Wall in an Applied Magnetic Field

Abstract: A numerical study is made of a moving magnetic domain wall. It is assumed that the wall moves with a uniform velocity V under the influence of an applied magnetic field of magnitude H_0. This leads to a boundary value problem on a doubly infinite line. By using a symmetry in the problem, the inherent difficulty of a two-dimensional search on a doubly infinite line is bypassed. For each V the problem is solved as a sequence of initial value problems involving a one-dimensional search. A limiting velocity is determined by means of an eigenvalue analysis. The curve representing the relation between V and H_0 is determined for a particular case.

The purpose of this paper is to make a numerical study of a moving 180° magnetic domain wall in a cubic crystal of composition $Ni_{0.75}\ Fe_{2.25}\ O_4$. The wall is assumed to have a uniform velocity of V (cm/sec) and to be moving under the influence of an applied magnetic field of magnitude H_0 (Oe). A formula, Eq. (4), is derived which represents the velocity V as a function of H_0 and of the wall shape. In Fig. 4 we give a curve representing the relation between V and H_0 for the particular case considered in this paper.

The magnetic vector for the domain wall is of uniform magnitude and its direction is a function of $\eta = z - Vt$. Fig. 1 indicates the orientation of the magnetic axes x, y, and z in terms of the axes x', y', and z' of the crystal, and Fig. 2 shows the angles ϕ and θ which define the direction of the magnetic vector for the domain wall. There are four easy directions in the xy plane. They are $\phi_0, \pi - \phi_0, \pi + \phi_0$, and $2\pi - \phi_0$ where $\phi_0 = \arctan \sqrt{2}$. On one side of the domain wall the magnetic vector for the crystal has the direction $\phi = \phi_0$, $\theta = 0$, and this is also the direction of the applied field. The easy direction $\phi = \pi - \phi_0$, $\theta = 0$ is achieved as an intermediate direction in the domain wall, and, finally, on the other side of the domain wall the direction is $\phi = \phi_0 + \pi$, $\theta = 0$.

By means of the system of torque equations, Eq. (5), the transition from $\phi = \phi_0$ to $\phi = \pi - \phi_0$ and the transition from $\phi = \pi - \phi_0$ to $\phi = \pi + \phi_0$ each define a boundary value problem on a doubly infinite line ($-\infty < \eta < \infty$). A symmetry is proved which reduces each problem to a boundary value problem on a half-infinite line ($0 \leq \eta < \infty$). An integral of the system of torque equations further reduces the problem so that each boundary value problem is solved through a sequence of initial value problems involving a search on the initial value of θ (i.e., the θ that corresponds to $\phi = \pi/2$ and $\phi = \pi$, respectively).

An eigenvalue-eigenvector analysis is made for the system of torque equations in the neighborhood of the equilibrium points, and this is used to show that there is an upper limit to the velocity that can be achieved in this model of domain wall motion. This analysis has also been utilized in a scheme for correcting the initial value of θ, $\theta(0)$. Since no satisfactory method was found for correcting $\theta(0)$ when V approached its upper limit, the relation between V and H_0 in the neighborhood of this limiting velocity is not determined in this paper.

Before presenting the system of torque equations, Eq. (5), we derive Eq. (4) which represents the relation between the velocity V and the magnitude of the applied field H_0. In 1950 Kittel[1] proposed that a magnetic domain wall under the influence of an applied field moves at such a velocity that the liberated magnetostatic energy is completely dissipated by the damping forces which resist the spin rotations occurring as the wall advances. Clogston[2] has shown that the power per unit area dissipated through the itinerant electron loss mechanism in a 180° domain wall in $Ni_{0.75}\ Fe_{2.25}\ O_4$ moving at low velocity V in a direction perpendicular to the (110) plane can be written

$$P = V^2 \tau \left(\frac{N}{kT}\right) \int_{\phi_0}^{\phi_0+\pi} 2W^2 \sin^2 \phi \cos^2 \phi \left(\frac{d\phi}{d\eta}\right) d\phi, \quad (1)$$

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where N is the number of mobile electrons per cm^3, W is a parameter associated with a representation of the energies of the mobile electrons, and τ is the electronic relaxation time. The rate at which magnetostatic energy is released is $2M_sH_0V$ per unit wall area, where H_0 is the effective applied field, i.e., the field above threshold, and M_s is the saturation magnetization. Setting these two quantities equal to each other, one obtains the following expression for the velocity of a domain wall:

$$V = \frac{2H_0M_s}{\tau\left(\frac{N}{kT}\right)\int_{\phi_0}^{\phi_0+\pi} 2W^2 \sin^2\phi \cos^2\phi\left(\frac{d\phi}{d\eta}\right) d\phi}. \qquad (2)$$

The parameters in Eq. (2) at 201 °K can be determined from existing experimental data, and the relationship between domain wall velocity and applied field can be verified by comparison with measurements made with picture frame samples at this temperature by Galt.[3] At very low velocities, one may use for $d\phi/d\eta$ the value for a domain wall at rest in the (110) plane. Representing the anisotropy energy by the first-order constant only, one may write $d\phi/d\eta$ as follows:[2]

$$d\phi/d\eta = (-K_1/12A)^{1/2}\,|2 - 3\sin^2\phi|,$$

where K_1 is the first-order anisotropy constant, and A is the exchange constant. The value of K_1 at 201 °K is -6.43×10^4 ergs/cm^3 [Ref. 4]; the value of A is 1.09×10^{-6} erg/cm [Ref. 3]. M_s is 330 gauss [Ref. 4].

The quantity $\tau NW^2/M_skT$ may be evaluated from ferrimagnetic resonance linewidth data. Clogston[2] has shown that for $Ni_{0.75}Fe_{2.25}O_4$ the contribution to the linewidth in the (110) plane produced by the itinerant electron loss mechanism is

$$\Delta H = \frac{1}{M_s}\frac{8}{3}\left(\frac{N}{kT}\right)W^2 \times \left(\frac{\sin^4\phi}{4} + \sin^2\phi\cos^2\phi\right)\frac{\omega\tau}{1+\omega^2\tau^2},$$

where ϕ is the direction of the d.c. field referred to the [100] axis, and ω is the angular velocity of the microwave field. Thus, we have

$$\Delta H_{[111]} - \Delta H_{[100]} = \frac{1}{M_s}\frac{8}{9}\left(\frac{N}{kT}\right)W^2\frac{\omega\tau}{1+\omega^2\tau^2}. \qquad (3)$$

Measurements of $\Delta H_{[111]} - \Delta H_{[100]}$ have been made at 24.0 GHz,[4] yielding a value of 190 gauss at 201 °K, and at 9.2 GHz,[5] yielding a value of 100 gauss.* Utilizing these

* An additional isotropic contribution to the linewidth is attributable to eddy currents. The effect of eddy current losses on domain wall velocities in this material is not significant. Cf. Ref. 3.

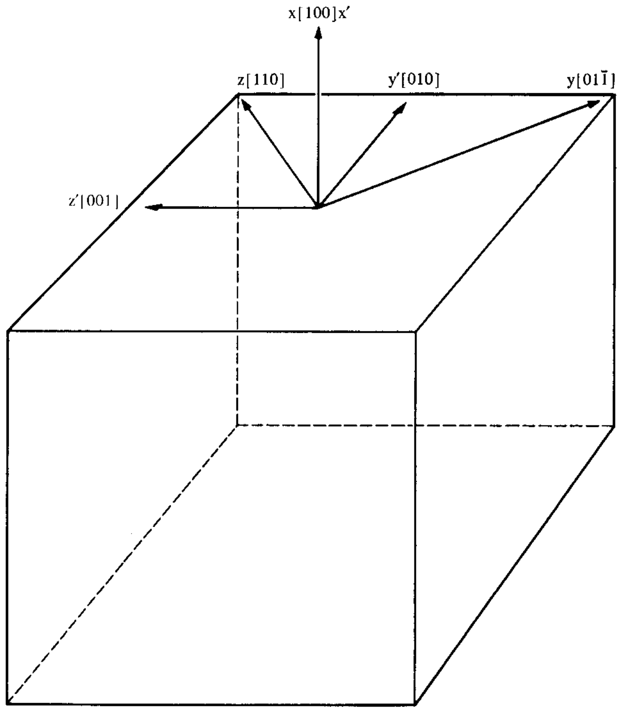
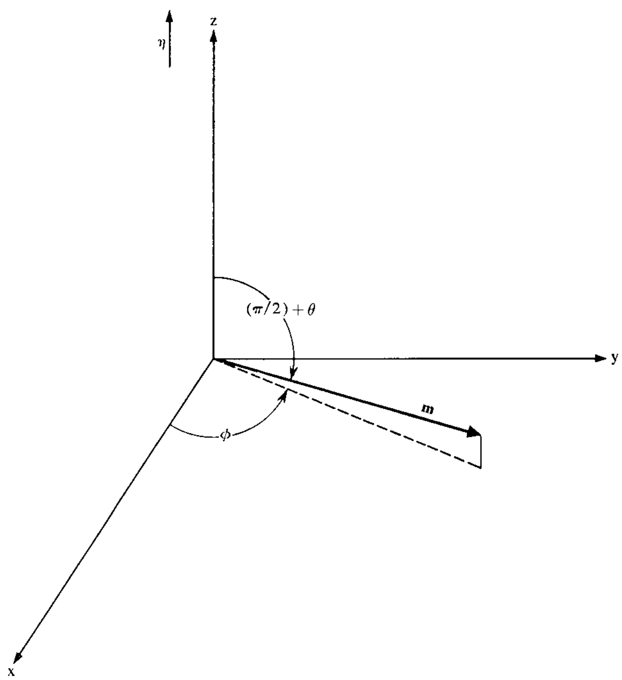


Figure 1 Orientation of the magnetic axes x, y, and z in terms of the crystal axes x', y', and z'.

Figure 2 Orientation of the angles ϕ and θ.



two results, we obtain a value for $(\tau N W^2)/(M_s kT)$ of 2.09×10^{-9} gauss/sec.

After the values of the parameters have been introduced in equation (2), the expression for the domain wall velocity is

$$V = 2H_0\left[2.09 \times 10^{-9} \text{ gauss/sec} \int_{\phi_0}^{\phi_0+\pi} 2 \sin^2 \phi \times \cos^2 \phi \frac{d\phi}{d\eta} d\phi\right]^{-1}$$

$$= (2.34 \times 10^4 H_0) \text{ cm/sec.}$$

This value is in fair agreement with the experimental value of the low velocity domain wall mobility of 26 150 cm/sec/gauss obtained by Galt.

The shape of a domain wall at rest is determined by the condition that the net torque acting on the spins within the wall vanishes at every point. This requires that the magnetization lie in the plane of the wall. For a wall in motion, on the other hand, the spins in the region of the wall must rotate between the directions of magnetization in the two adjacent domains, and the magnetization orientation assumed in the wall must generate a demagnetization field which will produce the required angular velocities[6,7]. In order that this field exist, the magnetization will not lie in the plane of the wall but will make some angle $\theta(\eta)$ with it. Clogston's calculation of the power dissipated by a moving wall whose "shape" $\phi(\eta)$ is that of a wall at rest can be extended to a wall of general shape $\phi(\eta)$, $\theta(\eta)$ by adding a term $\frac{2}{3} W^2 \sin^2\phi \, (d\theta/d\eta)^2/(d\phi/d\eta)^{-1}$ to the expression under the integral sign in Eq. (1) [Ref. 2]. Thus, we obtain

$$V = 2H_0 M\left\{2.09 \times 10^{-9} \text{ gauss/sec} \int_{\phi_0}^{\phi_0+\pi} 2 \sin^2 \phi \times \left[\cos^2 \phi\left(\frac{d\phi}{d\eta}\right) + \frac{1}{3}\left(\frac{d\theta}{d\eta}\right)^2\left(\frac{d\phi}{d\eta}\right)^{-1}\right] d\phi\right\}^{-1}. \quad (4)$$

In order to determine the wall shape for the particular example considered we start from the system of torque equations

$$-\frac{M_s V}{\gamma}\frac{d\mathbf{m}}{d\eta} = \mathbf{m} \times \left(2A\frac{d^2\mathbf{m}}{d\eta^2} - \frac{\partial E_a}{\partial \mathbf{m}} - 4\pi M_s^2 m_z i_z\right), \quad (5)$$

where $\mathbf{m} = (m_x, m_y, m_z)$ is the normalized magnetic vector $(m_x^2 + m_y^2 + m_z^2 = 1)$ and is assumed to be a function of $\eta = z - Vt$, $V =$ velocity in cm/sec of the wall. Also, $E_a =$ anisotropic energy $= K_1(\alpha_1^2\alpha_2^2 + \alpha_2^2\alpha_3^2 + \alpha_3^2\alpha_1^2)$, $\alpha_1 = m_x$, $\alpha_2 = (m_y + m_z)/\sqrt{2}$, $\alpha_3 = (m_z - m_y)/\sqrt{2}$,. $\gamma =$ gyromagnetic ratio $= 1.88 \times 10^7$ rad/sec/Oe. In terms of ϕ, θ we have $m_x = \cos\theta\cos\phi$, $m_y = \cos\theta \sin\phi$, $m_z = -\sin\theta$.

In the system of (5) we have included torques due to exchange, anisotropy, and internal (demagnetizing) fields. Our neglect of the applied field and of damping forces is justified when these fields are small compared with the others. Enz[8] has studied the dependence of wall shape (in the case of uniaxial anisotropy) as a function of wall velocity in the absence of applied and damping fields, and found a kind of "Lorentz contraction." We solve here the corresponding problem for the case of cubic anisotropy in the limit of weak damping.

The following pair of dimensionless equations is obtained from Eq. (5)

$$v\cos\theta\frac{d\theta}{d\mu} = \cos^2\theta\frac{d^2\phi}{d\mu^2} - 2\sin\theta\cos\theta\frac{d\phi}{d\mu}\frac{d\theta}{d\mu} + \epsilon\frac{\partial F}{\partial\phi}, \quad (6)$$

$$-v\cos\theta\frac{d\phi}{d\mu} = \frac{d^2\theta}{d\mu^2} + \sin\theta\cos\theta\left(\frac{d\phi}{d\mu}\right)^2 + \epsilon\frac{\partial F}{\partial\theta} - \sin\theta\cos\theta, \quad (7)$$

where quantities are as expressed in Eq. (8) to (12). [The term $-\frac{1}{3}$ is introduced in Eq. (8) so that $F = 0$ at its minima.]

$$F = -\tfrac{1}{3} + \tfrac{1}{4}[\cos^4\theta\sin^2\phi(3\cos^2\phi + 1) + 2\sin^2\theta\cos^2\theta(3\cos^2\phi - 1) + \sin^4\theta], \quad (8)$$

$$\mu = \zeta^{1/2}\eta, \quad (9)$$

$$v = \zeta^{-1/2}\left(\frac{M_s V}{2A\gamma}\right) = \frac{V}{\gamma(8\pi A)^{1/2}}, \quad (10)$$

$$\epsilon = -\zeta^{-1}\left(\frac{K_1}{2A}\right) = -\frac{K_1}{4\pi M_s^2}, \quad (11)$$

$$\zeta = \frac{2\pi M_s^2}{A}. \quad (12)$$

For the given values of the constant we have $\zeta = 6.28 \times 10^{11}$, $\mu = 7.92 \times 10^5\eta$, $\epsilon = 0.047$, and $v = 1.015 \times 10^{-5}V$. Multiplying Eq. (6) by $d\phi/d\mu$, Eq. (7) by $d\theta/d\mu$ and adding leads to the following integral of the system

$$I \equiv \cos^2\theta\left(\frac{d\phi}{d\mu}\right)^2 + \left(\frac{d\theta}{d\mu}\right)^2 + 2\epsilon F - \sin^2\theta. \quad (13)$$

In analyzing the system (6), (7) it is convenient to introduce the equivalent set of four first-order equations,

$$\frac{d\phi}{d\mu} = X, \quad (14)$$

$$\frac{d\theta}{d\mu} = Y, \quad (15)$$

$$\frac{dX}{d\mu} = -\epsilon g(\phi, \theta) + Y(2X\sin\theta + v)/\cos\theta, \qquad (16)$$

$$\frac{dY}{d\mu} = \sin\theta\cos\theta\,(1 - X^2 + \epsilon h(\phi, \theta)) - vX\cos\theta, \qquad (17)$$

where

$$g(\phi, \theta) = \sin\phi\cos\phi \times [\cos^2\theta(3\cos^2\phi - 1) - 3\sin^2\theta], \qquad (18)$$

$$h(\phi, \theta) = \cos^2\theta\sin^2\phi(3\cos^2\phi + 1) + (2\sin^2\theta - 1)(3\cos^2\phi - 1) - \sin^2\theta. \qquad (19)$$

The three equilibrium points of Eqs. (14)–(17) that we are concerned with are given by $P_i = (\phi_i, \theta_i, X_i, Y_i)$, $i = 1, 2, 3$, where $\theta_i = X_i = Y_i = 0, \phi_1 = \phi_0, \phi_2 = \pi - \phi_0, \phi_3 = \pi + \phi_0$, and $\phi_0 = \arctan\sqrt{2}$. These correspond to easy directions of magnetization, that is, to minima of E_a. Since $F = 0$ at these minima, then $I = 0$ at each of the equilibrium points.

What we seek for each value of V is a pair of trajectories $T_1(\mu)$ and $T_2(\mu)$ which are solutions to Eqs. (14)–(17). For T_1 we require that $T_1 \to P_1$ as $\mu \to -\infty$ and that $T_1 \to P_2$ as $\mu \to +\infty$. For T_2 we require that $T_2 \to P_2$ as $\mu \to -\infty$ and that $T_2 \to P_3$ as $\mu \to +\infty$. Once we have determined T_1 and T_2, we calculate the applied field (whose direction is $\phi = \phi_0$, $\theta = 0$, and whose magnitude is H_0 in oersteds) which will cause the domain wall to move with velocity V. H_0 is computed by the formula.

$$H_0 = 1.66 \times 10^{-3} V \int_{\phi_0}^{\phi_0+\pi} \sin^2\phi \times (X\cos^2\phi + \tfrac{1}{3}Y^2/X)\, d\phi. \qquad (20)$$

For values of $v \neq 0$ it is necessary to solve Eqs. (14)–(17) numerically. For T_1 we can specify that $\phi = \pi/2$ for $\mu = 0$. Because of Eq. (13) this leaves two initial values to be determined at $\mu = 0$ in such a way that the solution to Eqs. (14)–(17) will satisfy the boundary conditions at $\mu = \pm\infty$. The following theorem establishes a symmetry which enables us to avoid the difficulty of a two-dimensional search coupled with numerical integration on a doubly infinite line.

Theorem 1. If a trajectory T_1 exists which minimizes the free energy, Eq. (21), then $d\theta/d\mu = 0$ for $\phi = \pi/2$.

Before we prove this theorem we wish to examine its consequences. If we introduce $\psi = \phi - \pi/2$, then $F(\pi/2 + \psi, \theta) = F(\pi/2 - \psi, \theta)$ and hence $\partial F/\partial\psi$ is an odd function of ψ and $\partial F/\partial\theta$ is an even function of ψ. Also, $\psi(-\infty) = -(\pi/2 - \phi_0) = -\psi(\infty)$ and therefore the transformation $\psi \to -\psi$, $\mu \to -\mu$ leaves the Eqs. (6) and (7) and the boundary conditions invariant. Because the system is autonomous there is no loss of generality in specifying that for $\mu = 0, \psi = 0$ and hence by the theorem $d\theta/d\mu = 0$. If we have a solution $\bar\psi$, $\bar\theta$ to Eqs. (6), (7) for $\mu \geq 0$ satisfying the boundary conditions $\bar\psi(0) = 0$, $d\bar\theta/d\mu(0) = 0$, and $T_1 = P_2$ at $\mu = \infty$, then we have a solution for all μ with the desired boundary conditions by the extension

$$\psi(\mu) = \begin{cases} \bar\psi(\mu) & \mu \geq 0 \\ -\bar\psi(-\mu) & \mu \leq 0 \end{cases}$$

$$\theta(\mu) = \begin{cases} \bar\theta(\mu) & \mu \geq 0 \\ \bar\theta(-\mu) & \mu \leq 0. \end{cases}$$

By using Eq. (13), $x(0) > 0$ is determined in terms of $\theta(0)$, and, therefore, the above theorem reduces the problem of finding T_1 to a one-parameter search on the value of $\theta(0)$. For each $\theta(0)$ we have an initial value problem on the half line $\mu \geq 0$. Finding T_2 is similar except that $d\theta/d\mu = 0$ for $\phi = \pi$.

Proof of Theorem 1. With $\psi = \phi - \pi/2$, (6) and (7) are the Euler equations for the following variational problem for the free energy E.

$$\left.\begin{aligned} &E = \int_{-\infty}^{\infty} L\left(\psi, \theta, \frac{d\psi}{d\mu}, \frac{d\theta}{d\mu}\right) d\mu = \textit{minimum} \\ &T_1 = P_1 \text{ at } \mu = -\infty, \quad T_1 = P_2 \text{ at } \mu = +\infty \end{aligned}\right\} \qquad (21)$$

where

$$L = \frac{1}{2}\left[\left(\frac{d\psi}{d\mu}\right)^2\cos^2\theta + \left(\frac{d\theta}{d\mu}\right)^2 + \sin^2\theta\right] - v\frac{d\psi}{d\mu}\sin\theta - \epsilon F\left(\frac{\pi}{2} + \psi, \theta\right). \qquad (22)$$

Furthermore, we normalize μ such that $\mu = 0$ where $\psi = 0$. In the problem of (21) we admit functions θ whose derivative may have a jump discontinuity at the origin.

Lemma. The problem of (21) is equivalent to

$$\left.\begin{aligned} &J = \int_0^{\infty} L\, d\mu = \textit{minimum} \\ &T_1 = P_2 \quad \text{at} \quad \mu = \infty \quad \text{and} \quad \psi(0) = 0 \end{aligned}\right\}. \qquad (23)$$

Proof of the lemma

$$E = \int_{-\infty}^{\infty} L\, d\mu = \int_{-\infty}^{0} L\, d\mu + \int_0^{\infty} L\, d\mu = \hat{J} + J.$$

Let $\nu = -\mu$, $\psi(\mu) = -\hat\psi(\nu)$, $\theta(\mu) = \hat\theta(\nu)$ for $\mu \geq 0$. Then

$$\hat{J} = \int_{-\infty}^{0} L\left(\psi, \theta, \frac{d\psi}{d\mu}, \frac{d\theta}{d\mu}\right) d\mu = \int_0^{\infty} L\left(\hat\psi, \hat\theta, \frac{d\hat\psi}{d\nu}, \frac{d\hat\theta}{d\nu}\right) d\nu.$$

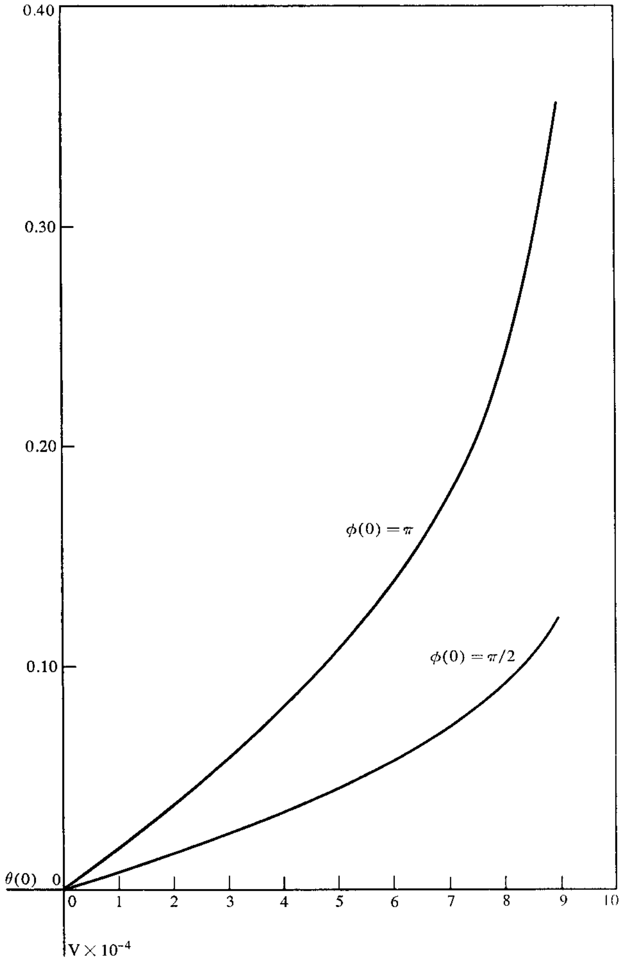


Figure 3 Velocity (cm/sec) of domain wall versus initial value of θ (radians).

Let ψ_1, θ_1 be a pair of trial functions for the problem of (21) and assume that $J_1 \leq \hat{J}_1$. We define a new pair of trial functions ψ_2, θ_2 such that

$$\psi_2(\mu) = \begin{cases} \psi_1(\mu) & \mu \geq 0 \\ -\psi_1(-\mu) & \mu \leq 0, \end{cases}$$

$$\theta_2(\mu) = \begin{cases} \theta_1(\mu) & \mu \geq 0 \\ \theta_1(-\mu) & \mu \leq 0. \end{cases}$$

Then $J_2 = \hat{J}_2 = J_1$ so $E_2 = 2J_1 \leq J_1 + \hat{J}_2 = E_1$. A similar symmetrizing reduction can be made if $\hat{J}_1 \leq J_1$. Since the problem is to minimize E, the trial functions for Eq. (21) can be restricted to odd functions for ψ and even functions for θ. In this way there is a one-to-one correspondence between the trial functions for the two problems. Moreover, in this class of functions

$E \equiv 2J$, and hence the equivalence is self-evident.

To complete the proof of Theorem 1 we note that the condition

$$\frac{d\theta}{d\mu} = L_{\theta'} = 0 \quad \text{at} \quad \mu = 0^+$$

is a natural boundary condition of (23) and necessarily must be satisfied for the minimum. Thus, for the solution to (21), $d\theta/d\mu$ is continuous at $\mu = 0$ and has the value zero. The proof for Theorem 1 is completed.

Theorem 2. Neither T_1 nor T_2 exist for

$$|v| \geq (1 + \delta)^{1/2} + (\delta)^{1/2}, \tag{24}$$

where

$$\delta = \tfrac{4}{3}\epsilon. \tag{25}$$

(We note that since $\epsilon = 0.047$, we must have $|v| < 1.29$; that is,

$$0 \leq V < 1.27 \times 10^5). \tag{26}$$

Proof: Let $\Phi = \phi - \phi_i$ where $i = 1$, 2, or 3. Then the linearized system for (14)–(17) about P_i is given by

$$\frac{d\Phi}{d\mu} = X, \tag{27}$$

$$\frac{d\theta}{d\mu} = Y, \tag{28}$$

$$\frac{dX}{d\mu} = \delta\Phi + vY, \tag{29}$$

$$\frac{dY}{d\mu} = (1 + \delta)\theta - vX. \tag{30}$$

The characteristic equation for the system is

$$\lambda^4 - (1 + 2\delta - v^2)\lambda^2 + \delta(1 + \delta) = 0. \tag{31}$$

The eigenvalues are $\pm\lambda_1$, $\pm\lambda_2$ where $\text{Re}(\lambda_j) \geq 0$,

$$\lambda_j^2 = \tfrac{1}{2}[1 + 2\delta - v^2 + (-1)^j\sqrt{D}] \tag{32}$$

and

$$D = (1 + 2\delta - v^2)^2 - 4\delta(1 + \delta). \tag{33}$$

If $|v| \geq (1 + \delta)^{\frac{1}{2}} + (\delta)^{\frac{1}{2}}$, then $0 \leq D < (1 + 2\delta - v^2)^2$ and $1 + 2\delta - v^2 < 0$ so $\lambda_j^2 < 0$. Thus, if Eq. (24) is satisfied, the eigenvalues for the linearized system are pure imaginary and no trajectory will approach the equilibrium points as $\mu \to \pm\infty$. *Q.E.D.*

From Eq. (32) and (33), we see also that for $|v| \leq (1 + \delta)^{\frac{1}{2}} - (\delta)^{\frac{1}{2}} = 0.781$ (i.e., $0 \leq V \leq 7.68 \times 10^4$), there are two positive and two negative real eigenvalues and for $(1 + \delta)^{\frac{1}{2}} - (\delta)^{\frac{1}{2}} < |v| < (1 + \delta)^{\frac{1}{2}} + (\delta)^{\frac{1}{2}}$, there are two eigenvalues with positive real parts and two with negative real parts.

If $\lambda_1 \neq \lambda_2$, the general solution to Eqs. (27)–(30) is

$$Q = A_1 U(\lambda_1)e^{\lambda_1\mu} + A_2 U(\lambda_2)e^{\lambda_2\mu} + B_1 U(-\lambda_1)e^{-\lambda_1\mu} + B_2 U(-\lambda_2)e^{-\lambda_2\mu} \quad (34)$$

where

$$Q = \begin{bmatrix} \Phi \\ \theta \\ X \\ Y \end{bmatrix} \quad (35)$$

and

$$U(\lambda) = \begin{bmatrix} 1 \\ \lambda \\ (\lambda^2 - \delta)/\lambda v \\ (\lambda^2 - \delta)/v \end{bmatrix}. \quad (36)$$

Consider, for example, the approach to the equilibrium point P_2 as $\mu \rightarrow +\infty$. In this case $\Phi = \phi - \phi_2$ where $\phi_2 = \pi - \arctan\sqrt{2}$. Since we want $Q \rightarrow 0$ as $\mu \rightarrow +\infty$, we must have $A_1 = A_2 = 0$. If the numerical integration of Eqs. (14)–(17) for a given value of $\theta(0)$ has proceeded to a value of μ such that

$$||Q|| = \Phi^2 + \theta^2 + X^2 + Y^2$$

is sufficiently small, then the linear system (27)–(30) is a valid approximation to (14)–(17) and we can solve the equations of (34) for A_1, A_2, B_1, B_2. The extent to which A_1 and A_2 are suppressed serves as a guide to the choice of $\theta(0)$. This method was used and numerical results were thereby obtained. Figure 3 is a graph of $\theta(0)$ as a function of V for $\phi(0) = \pi/2, \pi$ and $0 \leq V \leq 9 \times 10^4$. Figure 4 is a graph of H_0 vs. V for the same range of V.

For $V > 9 \times 10^4$ the approach to the equilibrium points is strongly oscillatory, and the use of the linear system did not prove adequate. The finding of an effective method of convergence for $\theta(0)$ remains open in the upper range of V.

If $0 < V < 7.68 \times 10^4$, then the eigenvalues are real and the approach to equilibrium is monotonic. In fact, θ, X, and Y can be taken as functions of ϕ so that (14)–(17) are replaced by

$$\frac{d\theta}{d\phi} = Y/X, \quad (37)$$

$$\frac{dX}{d\phi} = [-\epsilon g(\phi, \theta) + Y(2X\sin\theta + v)/\cos\theta]/X, \quad (38)$$

$$\frac{dY}{d\phi} = \sin\theta\cos\theta(1 - X^2 + \epsilon h(\phi, \theta))/X - v\cos\theta. \quad (39)$$

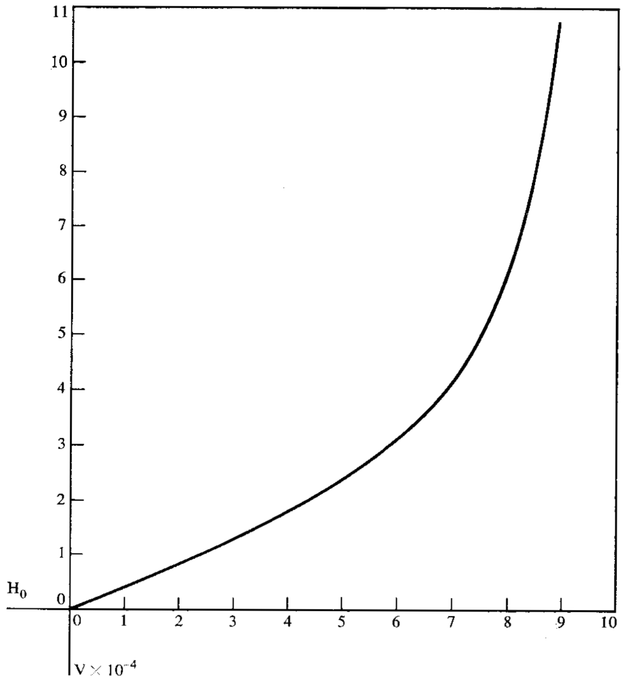


Figure 4 Velocity (cm/sec) of domain wall versus applied magnetic field (oersteds).

As before, consider $\phi \rightarrow \phi_2^-$ and suppose in (34) that $A_1 = A_2 = 0$. Since $0 < \lambda_1 < \lambda_2$, the term $B_1 u(-\lambda_1)e^{-\lambda_1\mu}$ dominates unless $B_1 = 0$. Thus, for $B_1 \neq 0$ we have a one-parameter family of trajectories such that

$$\frac{d\theta}{d\phi} = -\frac{\lambda_1^2 - \delta}{v\lambda_1}, \quad \frac{dX}{d\phi} = -\lambda_1, \quad \frac{dY}{d\phi} = \frac{\lambda_1^2 - \delta}{v}$$

at $\phi = \phi_2^-$. Now $0 < v < (\delta + 1)^{\frac{1}{2}} - \delta^{\frac{1}{2}} = 0.781$, so $v^2 < 1$ and

$$\lambda_1^2 - \delta = \tfrac{1}{2}[(1 - v^2) - D^{1/2}] = \frac{2\delta v^2}{(1 + v^2) + D^{1/2}} > 0.$$

Thus,

$$\theta > 0 \quad \text{as} \quad \phi \rightarrow \phi_2^-.$$

This suggests trying $\theta(0) > 0$, which proves to be correct. In the numerical calculations Eqs. (37)–(39) were used for $V \times 10^{-4} \leq 7.68$. In Figures 5, 6, and 7, respectively, θ, X, and Y are plotted as functions of ϕ for $V \times 10^{-4} = 0, 4, 6$, and 7.68.

Galt[3] has noted that the relationship between wall velocity and applied field does not remain linear at large fields, but takes on a curvature opposite to that of the calculated results shown in Figure 4. At the temperature of 201° K considered in this paper, the nonlinearity begins

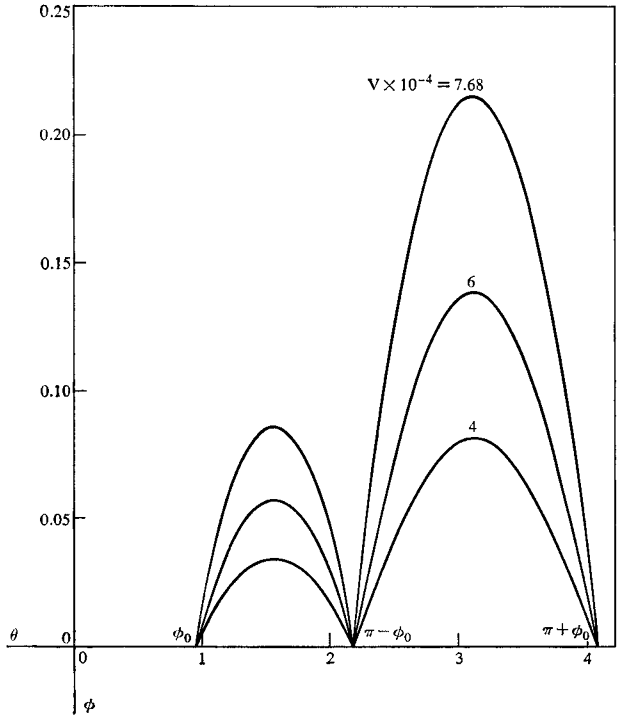
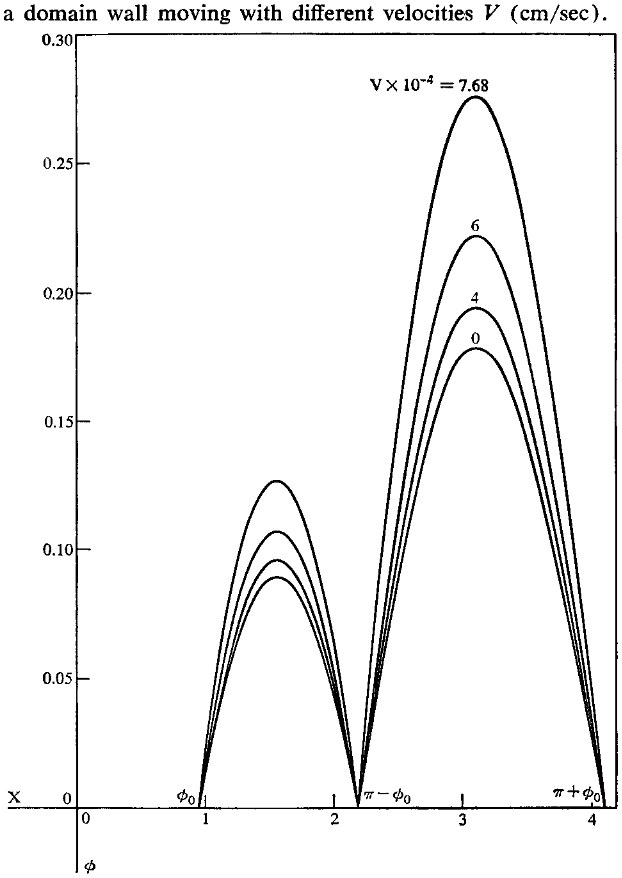


Figure 5 θ (radians) as a function of ϕ (radians) across a domain wall moving with different velocities V (cm/sec), $\theta \equiv 0$ for $V = 0$.

Figure 6 $X = d\phi/d\mu$ as a function of ϕ (radians) across a domain wall moving with different velocities V (cm/sec).



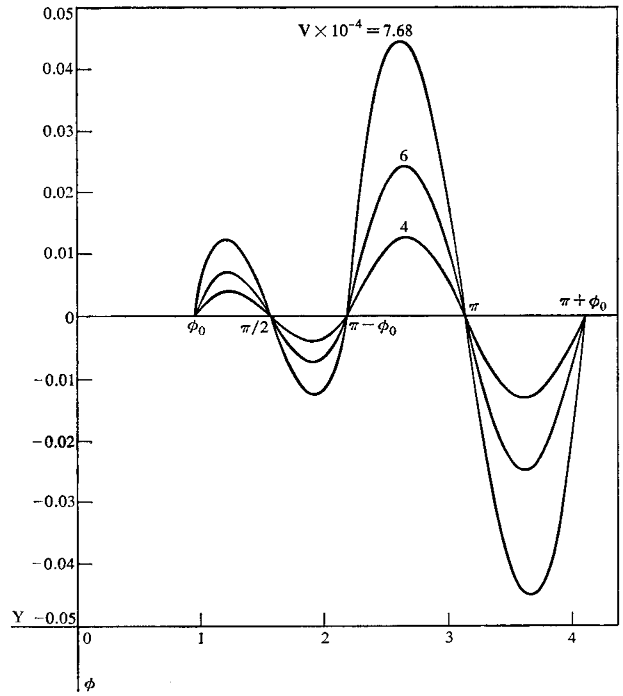


Figure 7 $Y = d\theta/d\mu$ as a function of ϕ (radians) across a domain wall moving with different velocities V (cm/sec). $Y \equiv 0$ for $V = 0$.

at a wall velocity of the order of 10^4 cm/sec, which is well inside the linear region of the calculated curve. Oscilloscope traces obtained by Galt indicate that in the nonlinear region the wall velocity becomes increasingly nonuniform as the field increases. This effect has not been explained. Consideration of nonuniform wall velocities and variable wall shapes is outside the scope of the time-independent treatment presented here, but such possibilities at higher velocities merit study.

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