W. E. Dickinson
R. M. Walker

Reliability Improvement by the Use of Multiple-Element Switching Circuits

Abstract: Physical devices used for switching have finite probabilities of failure. Circuits which make use of redundancy to achieve resultant reliabilities greater than that of their elements have been proposed and have been analyzed for the case of intermittent failures. The present paper extends certain of these results to the case of permanent failures of the elements, assuming that the reliability of these elements is known. It is shown that, for operating periods which are short compared to the mean time to failure of the elements, a substantial increase in reliability can be obtained by such redundancy.

The problem of constructing more reliable switching circuits by the use of redundant elements has been considered by von Neumann[1] and by Moore and Shannon[2] for certain cases of intermittent failures of elements whose probability of failure is constant with time. It is the purpose of this paper to point out that their results can be extended to cover cases of permanent failures of elements, provided that the reliability* of the elements is known. The examples used for illustration are circuit arrangements suggested in the reference papers.

The assumptions used in those papers were as follows:

1. The probability of failure of any element is independent of the probability of failure of any other element.
2. Only intermittent failures are considered.
3. The probability of failure of an element is defined for each operation, is constant with time, and is the same for every element.

In this paper, we assume the following:

1. The probability of failure of any element is independent of the probability of failure of any other element.
2. All failures are permanent; that is, when an element fails, it remains in the failed condition.
3. The reliability of the elements is known (as a function of time) and is the same for every element.

We define the following probabilities:

$F(t)$ = the probability that an element or circuit will fail during the time interval 0 to t. This is the cumulative distribution function of time to failure.

$R(t) = 1 - F(t)$ = the probability that an element or circuit will not fail during the time interval 0 to t. This is the reliability for the time interval 0 to t.

$Z(t)dt$ = the probability that an element or circuit will fail during the infinitesimal interval dt beginning at time t, conditional on non-failure prior to time t. $Z(t)$ may then be called the failure rate for survivors. (We will assume that $Z(t)$ may be any non-negative function of t which is integrable over any finite time interval; it need not be continuous. We also assume that $Z(t)$ does not approach zero as $t \rightarrow \infty$.)

The relation between $Z(t)$ and $R(t)$ is:

$$R(t+dt) = R(t) - R(t)Z(t)dt, \tag{1}$$

$$\frac{dR(t)}{R(t)} = -Z(t)dt, \tag{2}$$

$$R(t) = \exp\left[-\int_0^t Z(\tau)d\tau\right]. \tag{3}$$

If $Z(t) = \lambda$, where λ is a constant, the reliability becomes

$$R_e(t) = e^{-\lambda t} \tag{4}$$

which is the well-known exponential law of failure.[3,4]

The cumulative failure distribution function for an element obeying the exponential law is then

$$F_e(t) = 1 - R_e(t) = 1 - e^{-\lambda t}. \tag{5}$$

This is shown as curve 0 of Fig. 4 and is plotted using normalized time, λt.

With the preceding assumptions regarding the reliability of the elements we now consider two types of multiple-element circuits.

The von Neumann 2-out-of-3 majority[1]

Von Neumann suggested the use of a majority organ fed by three independent logical devices (hereafter referred to as elements) which operate from the same source of

---

*Reliability as used herein means the probability of adequate performance of the prescribed function for a specified time.

input information, as shown in Fig. 1. We shall assume that the majority organ itself is perfect and that the only failures to be considered are those of the three elements which feed it. These may fail in two modes; (a) with the output remaining permanently in the "low" voltage state, or (b) with the output remaining permanently in the "high" voltage state. We investigate two cases having different assumptions in this respect.

- *Case 1*

This is the upper bound for failure of this arrangement, obtained by assuming that all failures of the elements occur in the same mode. Then failure of the redundant system will occur if any two or three of the elements fail, or conversely, the system will not fail if all three or any two of the elements still function correctly. If the probability that any element will function correctly is R_o, the probability that all three elements will function correctly is R_o^3; the probability that any one element fails and the other two still function correctly is $(1-R_o)R_o^2$. Since this latter possibility can happen in three different ways the reliability, i.e., the probability that the redundant system will not fail, will be

$$R_1 = R_o^3 + 3(1-R_o)R_o^2 = 3R_o^2 - 2R_o^3. \qquad (6)$$

R_1 is plotted against R_o as curve 1 of Figure 3. This relation is valid for any $R_o(t)$. However, if the exponential law is assumed for the elements, then by substituting $R_e(t)$ for R_o we obtain

$$R_{1_e} = 3e^{-2\lambda t} - 2e^{-3\lambda t}. \qquad (7)$$

The corresponding cumulative failure distribution function, $F_{1_e} = 1 - R_{1_e}$, is shown as curve 1 of Figure 4.

- *Case 2*

For this we assume that the elements have equal probability of failing in the "high" or the "low" state. This means that when exactly two elements fail, there is a probability of 1/2 that they will fail in opposite directions, in which case the output of the system would still be correct. The probability that two elements fail and the other still functions correctly is $(1-R_o)^2 R_o$, and this can happen in three different ways. Therefore, the reliability

$$R_2 = R_o^3 + 3(1-R_o)R_o^2 + \frac{3}{2}(1-R_o)^2 R_o$$

$$= \frac{3}{2}R_o - \frac{1}{2}R_o^3. \qquad (8)$$

This is plotted vs. R_o as curve 2 of Fig. 3. Since $0 \le R_o \le 1$, then $R_o^3 \le R_o$, and it follows that $R_2 \ge R_o$ for any R_o. Also, for the exponential law,

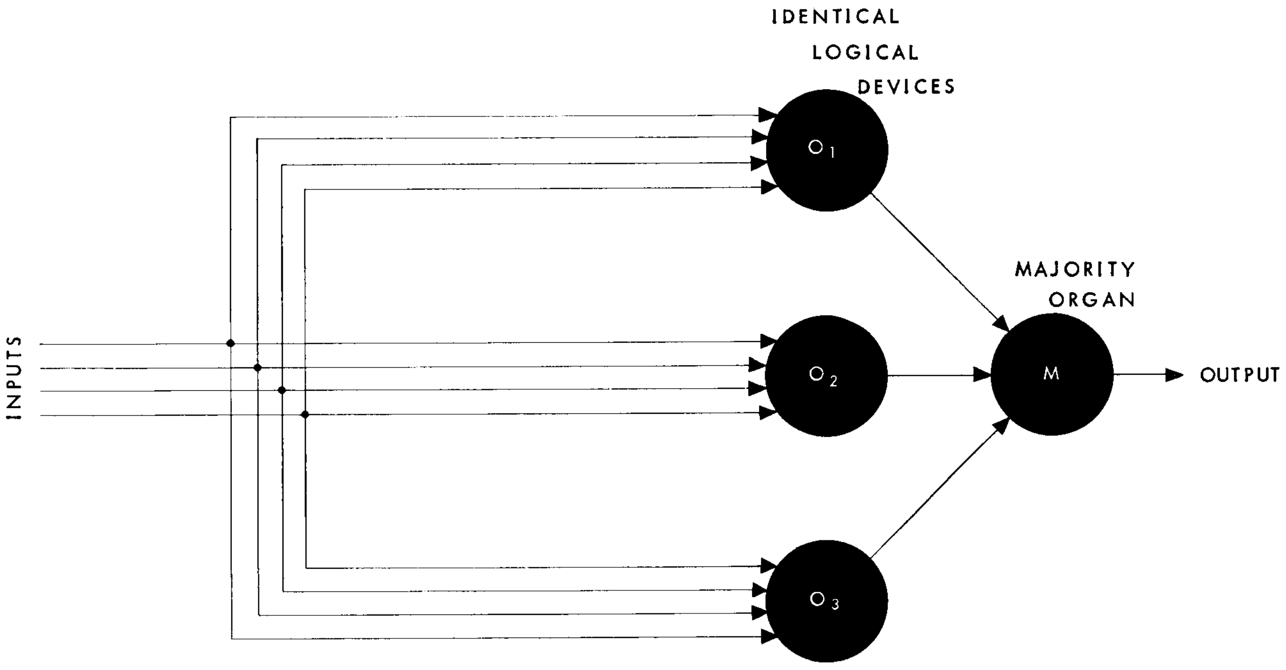
$$R_{2_e} = \frac{3}{2}e^{-\lambda t} - \frac{1}{2}e^{-3\lambda t}. \qquad (9)$$

The corresponding $F_{2_e} = 1 - R_{2_e}$, is shown as curve 2 of Fig. 4.

The Shannon-Moore series-parallel relay contact network[2]

This redundant circuit uses four identical relays, each

Figure 1 **Two-out-of-three majority circuit.**



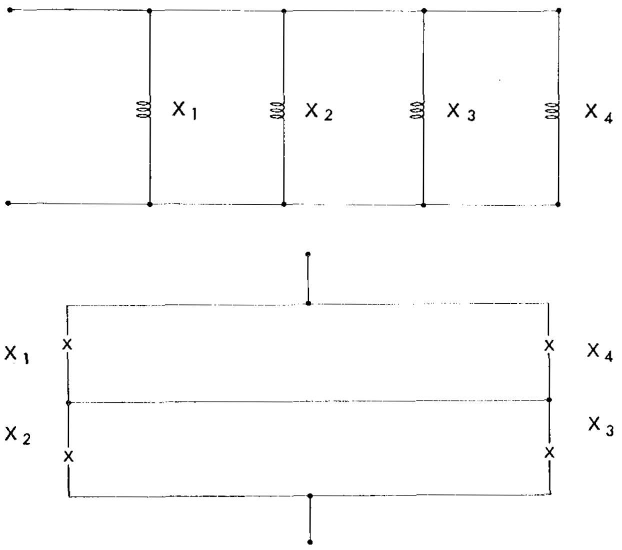


Figure 2 **Series-parallel relay contact network on four relays.**

with one contact, to replace a single relay having one contact. The arrangement is as shown in Fig. 2. The four coils in parallel replace the single coil and the contact network shown replaces the single contact. Two such contact arrangements were discussed by Shannon and Moore, with and without the center connection; we shall consider only this one.

Shannon and Moore have shown that for a two-terminal network made up of m statistically independent contacts, each contact of which has the probability p of being closed, the network will have a probability $h(p)$ of being closed, as follows:

$$h(p) = \sum_{n=0}^{m} A_n p^n (1-p)^{m-n}, \tag{10}$$

where A_n is the number of ways we can select a subset of n contacts such that if these n contacts are closed and all others open, then the network will be closed.

Similarly, the probability of the network being open is

$$1-h(p) = \sum_{n=0}^{m} B_n (1-p)^n p^{m-n}, \tag{11}$$

where B_n is the number of subsets of n contacts such that if all contacts in a subset are open and all others closed, the network is open.

Neglecting the possibility of coil failures, each contact may fail in two ways: (a) it may fail to make contact when it should, or (b) it may fail to break contact when it should. As used herein, the term "fail to make" means failure of a contact or a circuit to be closed whenever it should be closed, and "fail to break" means failure of a contact or circuit to be open whenever it should be open. As previously stated, either type of failure is assumed to be permanent.

We define two cumulative failure distributions:

$q_a(t)$ = the probability that a contact will fail to make during the interval 0 to t.

$p_b(t)$ = the probability that a contact will fail to break during the interval 0 to t.

It follows that:

$p_a(t) = 1 - q_a(t)$ = the probability that a contact will be closed whenever it should be closed during the interval 0 to t. This is the reliability of being closed, defined for this interval.

$q_b(t) = 1 - p_b(t)$ = the probability that a contact will be open whenever it should be open during the interval 0 to t. This is the reliability of being open, defined for this interval.

The total probability of failure of the circuit in the interval 0 to t is the sum of the disjoint probabilities of failure to make and failure to break. The probability that this circuit will fail to make at some time during the interval 0 to t is

$$1-h(p_a) = \sum_{n=0}^{4} B_n (1-p_a)^n p_a^{4-n} = \sum_{n=0}^{4} B_n q_a^n (1-q_a)^{4-n}. \tag{12a}$$

Since $B_0 = B_1 = 0$, $B_2 = 2$, $B_3 = 4$, $B_4 = 1$,

$$\begin{aligned} 1-h(p_a) &= 2q_a^2(1-q_a)^2 + 4q_a^3(1-q_a) + q_a^4 \\ &= 2q_a^2 - q_a^4. \end{aligned} \tag{12b}$$

Similarly, the probability of the circuit failing to break at some time during the interval 0 to t is

$$\begin{aligned} h(p_b) &= \sum_{n=0}^{4} A_n p_b^n (1-p_b)^{4-n} \\ &= 4p_b^2(1-p_b)^2 + 4p_b^3(1-p_b) + p_b^4 \\ &= 4p_b^2 - 4p_b^3 + p_b^4. \end{aligned} \tag{13}$$

Then the total probability of circuit failure in the interval 0 to t is

$$F_r = 2q_a^2 - q_a^4 + 4p_b^2 - 4p_b^3 + p_b^4. \tag{14}$$

We shall now consider three cases for this circuit with different assumptions as to failure probabilities. In all these cases F_o is the total probability that any given contact will fail.

● *Case 3*

All failures are assumed to be failures to make. Then

$q_a = F_o$, $p_b = 0$, and

$$F_3 = 2F_o^2 - F_o^4, \tag{15}$$

or

$$\begin{aligned} R_3 &= 1 - [2(1-R_o)^2 - (1-R_o)^4] \\ &= 4R_o^2 - 4R_o^3 + R_o^4. \end{aligned} \tag{16}$$

This is shown as curve 3 of Fig. 3. For the exponential law,

$$R_{3_e} = 4e^{-2\lambda t} - 4e^{-3\lambda t} + e^{-4\lambda t}. \tag{17}$$

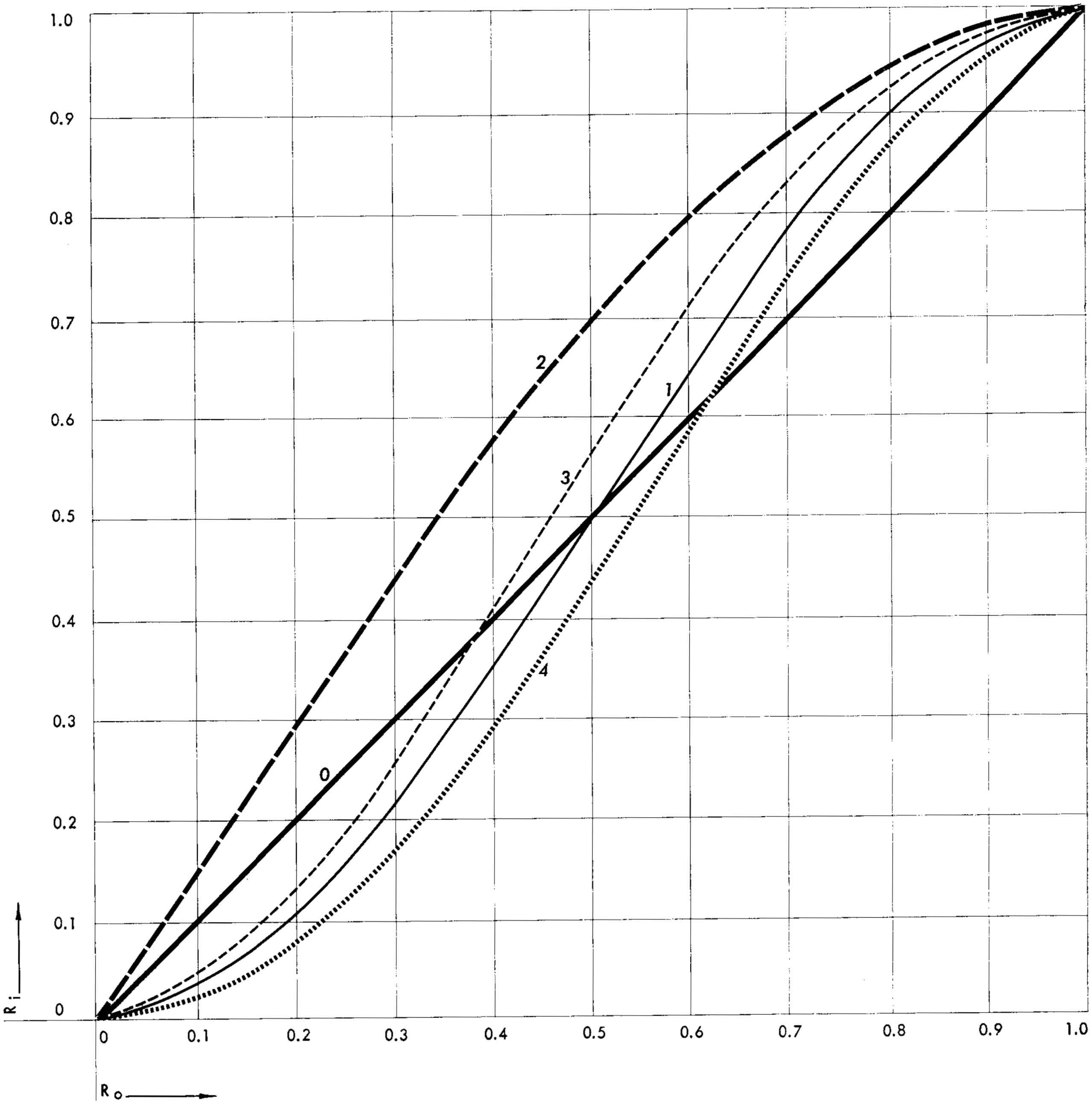


Figure 3 **Circuit reliability versus element reliability.**

The corresponding $F_{3_e} = 1 - R_{3_e}$ is shown as curve 3 of Fig. 4.

- *Case 4*

All failures are assumed to be due to contacts failing to break. Then $q_a = 0$, $p_b = F_o$, and

$$F_4 = 4F_o^2 - 4F_o^3 + F_o^4, \tag{18}$$

or

$$R_4 = 1 - F_4 = 1 - [4(1-R_o)^2 - 4(1-R_o^3) + (1-R_o)^4]$$

$$= 2R_o^2 - R_o^4. \tag{19}$$

This is plotted as curve 4 of Fig. 3. For the exponential law,

$$R_{4_e} = 2e^{-2\lambda t} - e^{-4\lambda t}, \tag{20}$$

and the corresponding $F_{4_e} = 1 - R_{4_e}$ is shown as curve 4 of Fig. 4.

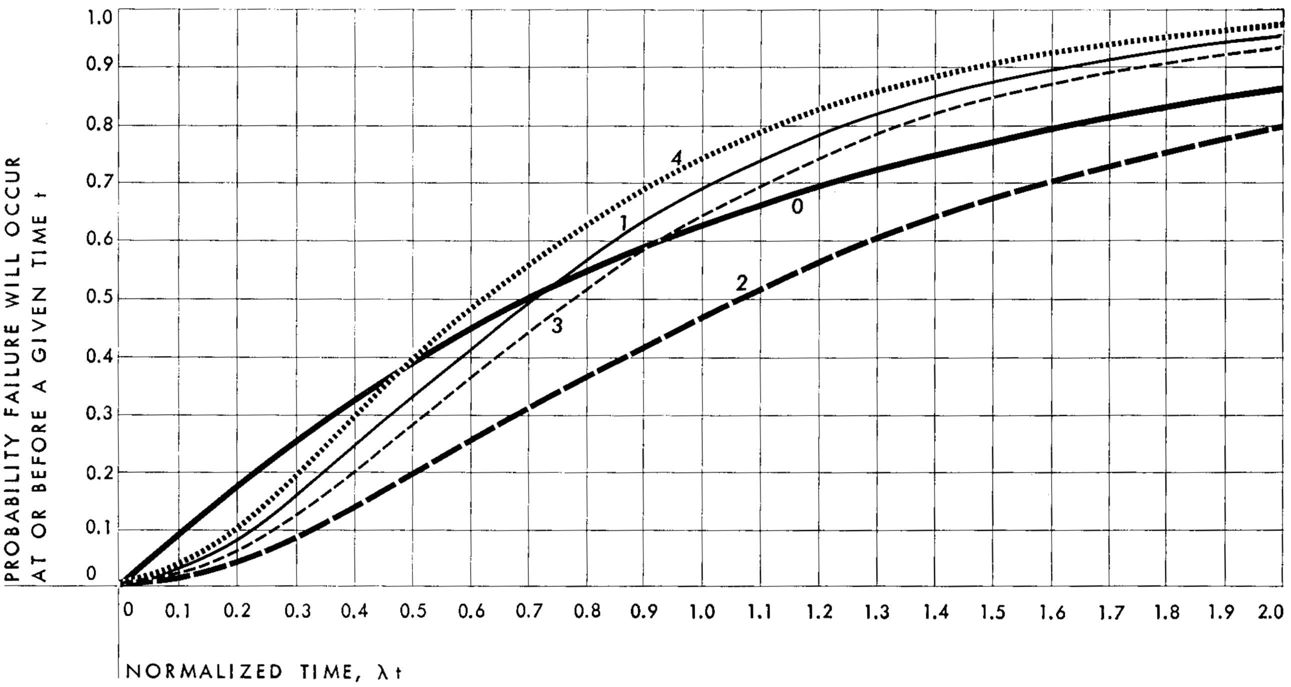


Figure 4 **Probability of circuit failure versus time for elements obeying exponential law.**

● *Case 5*

Failures to make and failures to break are considered equally probable. Then $q_a = p_b = \frac{1}{2}F_o$, and

$$F_5 = 2\left(\frac{F_o}{2}\right)^2 - \left(\frac{F_o}{2}\right)^4 + 4\left(\frac{F_o}{2}\right)^2 - 4\left(\frac{F_o}{2}\right)^3 + \left(\frac{F_o}{2}\right)^4$$

$$= \frac{3}{2}F_o^2 - \frac{1}{2}F_o^3, \qquad (21)$$

or

$$R_5 = 1 - \left[\frac{3}{2}(1-R_o)^2 - \frac{1}{2}(1-R_o)^3\right]$$

$$= \frac{3}{2}R_o - \frac{1}{2}R_o^3. \qquad (22)$$

Since R_5 is identical with R_2, it also is represented by curve 2 of Fig. 3. And similarly, for the exponential law, $F_{5_e} = F_{2_e}$, and is represented by curve 2 of Fig. 4. As for Case 2, we note that for any R_o, $R_5 \geq R_o$.

Figure 3 shows that for Cases 1, 3 and 4 the use of redundancy may decrease the reliability of the circuit to below that of the elements, and that the circuit is more reliable than its elements only if the reliability of these elements exceeds some minimum value for each case. However, for Cases 2 and 5, the redundant circuit is always more reliable, regardless of the element reliability.

Figure 4 shows that if the exponential law of reliability is applicable to the elements, then the reliability improvement for these circuits (as indicated by a reduced probability of failure) is most effective when the desired period of operation is such that $\lambda t << 1$. Since (see the next section) $\lambda = 1/m_o$, where m_o is the mean time to failure for an element, this means that the significant gain in reliability is obtained for $t << m_o$.

Mean time to failure

If no maintenance of the equipment is contemplated except when failure of the redundant circuit occurs, then the figure of interest is the mean time to failure, m.

Equation (3) may be rewritten as

$$R(t) = 1 - F(t) = e^{-\gamma(t)}, \qquad (23)$$

where

$$\gamma(t) = \int_0^t Z(\tau)\,d\tau, \qquad \text{with } \gamma(0) = 0. \qquad (24)$$

The mean time to failure is

$$m = \int_0^1 t\,dF = \int_0^\infty t e^{-\gamma(t)} \frac{d\gamma}{dt}\,dt$$

$$= -\left[te^{-\gamma}\right]_0^\infty + \int_0^\infty e^{-\gamma}dt. \qquad (25)$$

But

$$\lim_{t\to\infty}\left[te^{-\gamma}\right] = \lim_{t\to\infty}\left[\frac{t}{e^\gamma}\right] = \lim_{t\to\infty}\left[\frac{1}{e^\gamma \cdot Z(t)}\right].$$

Since $Z(t)$ does not approach zero as $t \rightarrow \infty$, then $e^{\gamma(t)}Z(t) \rightarrow \infty$ as $t \rightarrow \infty$, and therefore,

$$m = \int_0^\infty e^{-\gamma(t)} dt = \int_0^\infty R(t)\, dt. \tag{26}$$

For the exponential law, where it is assumed that the element reliability is $R_o = e^{-\lambda t}$, then

$$m_o = \frac{1}{\lambda}. \tag{27}$$

With this element reliability, the mean times to failure for the five cases are as follows:

$$m_1 = \frac{3}{2\lambda} - \frac{2}{3\lambda} = \frac{5}{6} m_o; \tag{28}$$

$$m_2 = \frac{3}{2\lambda} - \frac{1}{6\lambda} = \frac{4}{3} m_o; \tag{29}$$

$$m_3 = \frac{4}{2\lambda} - \frac{4}{3\lambda} + \frac{1}{4\lambda} = \frac{11}{12} m_o; \tag{30}$$

$$m_4 = \frac{2}{2\lambda} - \frac{1}{4\lambda} = \frac{3}{4} m_o; \tag{31}$$

$$m_5 = m_2 = \frac{4}{3} m_o. \tag{32}$$

Conclusion

Since, for any of these cases, the reliability of the redundant circuit is not linearly related to the reliability of its elements, it follows that any relative comparison must be made on the basis of particular assumptions about the operating time, the preventive maintenance procedures, and the failure distribution.

For unattended operation until failure, without preventive maintenance, and assuming the exponential law applies to the elements, circuits of either type can give only a 4/3 gain in mean time to failure for the most favorable assumption about failure modes (Cases 2 and 5).

However, if the desired operating period between maintenance intervals is short compared to m_o, a very substantial gain is possible for either circuit arrangement and for any of these assumptions about failure modes. For example (again assuming the exponential law), if one had elements with a mean life of 10,000 hours, their reliability for a 100-hour operating period would be predicted to be 0.99, i.e., the probability of failure of an element during that period would be 0.01. For Cases 2 or 5, the resultant probability of failure of the redundant circuit during 100 hours would be 0.00015, or about 1/67 as much as for the single element. If all elements were checked after the 100-hour operating period and all failed elements replaced, then, since the exponential law implies the absence of wear-out effects, the same reliability could be expected for the next 100 hours of operation.

References

1. J. von Neumann, "Probabalistic Logics," *Automata Studies*, Princeton Univ. Press, 1956.
2. E. F. Moore and C. Shannon, "Reliable Circuits Using Less Reliable Relays," *Jour. Franklin Inst.*, Sept.-Oct., 1956.
3. D. J. Davis, "An Analysis of Some Failure Data," Rand Corporation Report P183, February 12, 1952.
4. "Glossary and Dictionary of Terms and Definitions Relating Specifically to Reliability," collected and edited by C. M. Ryerson, Third National Symposium for Reliability and Quality Control, Washington, D. C., January 15, 1957.

Received December 23, 1957