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INSPECTION METHODS IN PROGRAMMING

CHARLES RICH

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by

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The Artificial Intelligence Laboratory
Massachusetts Institute of Technology

June 1981

A revised version of thesis submitted to the Department of Electrical Engineering and Computer Science on May 16, 1980 in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

This report describes research done at the Artificial Intelligence Laboratory of the Massachusetts Institute of Technology. Support for the laboratory's artificial intelligence research has been provided in part by the Advanced Research Projects Agency of the Department of Defense under Office of Naval Research contracts N00014-75-C-0643 and N00014-80-C-0505, and in part by National Science Foundation grant MCS-7912179.

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The work reported here lies in the area of overlap between artificial intelligence and software engineering. As research in artificial intelligence, it is a step towards a model of problem solving in the domain of programming. In particular, this work focuses on the routine aspects of programming which involve the application of previous experience with similar programs. I call this programming *by inspection.*

Programming is viewed here as a kind of engineering activity. Analysis and synthesis by inspection

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) are a prominent part of expert problem solving in many other engineering disciplines, such as electrical and mechanical engineering. The notion of inspection methods in programming developed in this work is motivated by similar notions in other areas of engineering.

This work is also motivated by current practical concerns in the area of software engineering. The inadequacy of current programming technology is universally recognized. Part of the solution to this problem will be to increase the level of automation in programming. I believe that the next major step in the evolution of more automated programming will be interactive systems which provide a mixture of partially automated program analysis, synthesis and verification. One such system being developed at MIT, called the *programmer's apprentice*, is the immediate intended application of this work.

This report concentrates on the knowledge base of the programmer's apprentice, which is in the form of a *taxonomy* of commonly used algorithms and data structures. To the extent that a programmer is able to construct and manipulate programs in terms of the forms in such a taxonomy, he may relieve himself of many details and generally raise the conceptual level of his interaction with the system, as compared with present day programming environments. Also, since it is practical to expend a great deal of effort pre-analyzing the entries in a library, the difficulty of verifying the correctness of programs constructed this way is correspondingly reduced. The feasibility of this approach is demonstrated by the design of an initial library of common techniques for manipulating symbolic data.

This document also reports on the further development of a formalism called the *plan calculus* for specifying computations in a programming language independent manner. This formalism combines both data and control abstraction in a uniform framework and has facilities for representing multiple points of view and side effects.

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Abstract

The work reported here lies in the area of overlap between artificial intelligence and software engineering. As research in artificial intelligence, it is a step towards a model of problem solving in the domain of programming. In particular, this work focuses on the routine aspects of programming which involve the application of previous experience with similar programs. I call this programming *by inspection.*

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This document also reports on the further development of a formalism called the *plan calculus* for specifying computations in a programming language independent manner. This formalism combines both data and control abstraction in a uniform framework and has facilities for representing multiple points of view and side effects.

To My Father.

It is not upon you to complete the task,
But neither may you shy away from its undertaking.

Pirke Avot

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PART III - TECHNICAL DETAILS

CHAPTER ONE
INTRODUCTION

1.1 Inspection Methods

Inspection methods are a distillation of the collective experience of solving many problems in a particular domain. The essence of this experience is a taxonomy of common problem *forms*. The first step of any inspection method is to *recognize* a familiar form embedded in a given problem. Associated with each such problem form is either an explicit solution or, more generally, the form of the answer. In sufficiently complex situations, debugging is also an unavoidable part of the use of inspection methods. The role of debugging in problem solving has been investigated by Sussman [68,67]; it is not part of the focus of this work.

For example, analysis of the termination conditions of a program is often done by inspection. If you recognize a loop that counts up by one from an initial number up to a fixed greater number, then you know from experience that it always terminates. Similarly, experienced programmers know a repertoire of standard operations on sets and their implementations for variety of set representations. In synthesis by inspection, once a programmer recognizes that a problem calls for one of these operations, he can implement it immediately. Program verification can also often done by inspection. Most of the difficult deductive steps (typically the inductive arguments) can be embedded in pre-proven lemmas which are associated with the standard forms. All that remains is to combine these lemmas appropriately in the proof of the particular program.

An Engineering Vocabulary

Another significant characteristic of the use of inspection methods in engineering is that the common forms acquire names which become part of the standard working vocabulary of experts in the field. These names for intermediate level constructs supplement the primitive vocabulary of the domain. For example, the primitive vocabulary of currents, voltages and resistances is formally adequate for specifying a wide range of electrical functions. Experienced electrical engineers, however, use a much richer vocabulary including such concepts as series and parallel configuration, voltage divider, cascode connection, and so on. Similarly, an experienced programmer knows much more than the the primitive programming language constructs, such as tests, iterations, arrays, assignments, and so on. An experienced programmer is also familiar with many other more abstract concepts such as lists, hash tables, search loops, and splicing.

A shared intermediate level vocabulary is very important for communication between experts. In many fields this vocabulary has been codified and is taught as part of the standard education of novices. This implies that facility with the appropriate intermediate vocabulary is an essential component of an intelligent interactive system which is going to help experts in some field. Chapter Two illustrates this point for the programmer's apprentice system in particular.

Uniform General Methods

Many areas of engineering (and related fields such as applied mathematics) have over a period of time developed powerful general methods which solve a wide range of problems of a given kind. For example, general circuit analysis techniques involving node and cut sets and the inversion of matrices have been known for a long time. Recently, a very powerful general method for symbolic integration has been discovered by Risch. Why then do inspection methods continue to be of interest?

General methods gain their power by operating in a uniform way at the most primitive level of vocabulary of the domain. This causes two serious problems: the methods are inefficient and the results are difficult for users to interpret. For example, the Risch algorithm is usually used only as a last resort, even by automated systems like Macsyma [42], because inspecting an integral for one of the many well-known forms is comparatively inexpensive, and if one is recognized, the answer can be computed much more quickly than by the algorithm. Similarly, general circuit analysis techniques involving node and cut sets and the inversion of matrices are seldom employed by expert circuit designers because they are so laborious in comparison to decomposing a circuit into familiar patterns with known behavior forms. Furthermore the decomposition into standard forms usually coincides with the modules of the design being explored.

Because of these difficulties, experts tend to employ uniform general methods only as a last resort. Whenever possible they try to work with familiar special cases which can be solved by inspection. In fact, this behavior is usually taken as one of the distinguishing characteristics of being an expert.

General methods have recently been developed in the area of programming also. For example, a general method for program verification due originally to Floyd [26] and Hoare [35] decomposes the problem into two steps. The first step is the generation of verification conditions, in which specifications of the desired behavior of the program are combined with the axioms for each language primitive in the program, yielding a single formula to be proved valid. This formula is then passed to a general purpose theorem prover. Unfortunately, if the program is incorrect, which is the most common case, the manner in which the proof of the verification conditions fails provides little guidance to the user about how to correct the original program. Verification by inspection, while it is not as powerful, does not suffer from this problem of incomprehensibility. Errors are detected by inspection either by recognizing a known pattern whose pre-proven properties contradict the desired specifications, or by recognizing a suspiciously close match to a known pattern. In either case, the nature of the discrepancy can be communicated to the user in terms of familiar engineering vocabulary.

The analysis of programs with side effects is another area in which general methods have failed to supplant inspection. Some work has been done on representing and reasoning about side effects in programs [64], but the general methods developed thus far are clumsy and computationally expensive. Furthermore, there is reason to believe that there are fundamental limitations to the effectiveness of general methods in this area. Programs with an unconstrained use of side effects (such as RPLACA and RPLACD in Lisp) are extremely difficult to understand even for the most expert human programmers. This has led some to advocate the extreme position of banning side effects entirely in new languages and systems. However, there are also good arguments that side effects are crucial for the modularity and efficiency of certain programs [66]. The resolution of this apparent conflict lies in the observation that side effects are typically used only in very stylized forms, such as to splice nodes in and out of a linked list, to update a global data base, and so on. By constructing a library of these standard plans and their properties, analysis of side effects by inspection can suffice for most practical purposes.

Education

The importance of inspection methods in engineering problem solving is also reflected in educational practices. The introductory parts of most engineering curricula first acquaint students with the standard forms of the discipline. Only much later, after the students' intuitions are developed, are the uniform general methods taught. For example, electrical engineering students are first taught how to predict the behavior of certain standard circuits (e.g. oscillators), and how to implement certain common signal processing functions (e.g. filters), before they are taught general tools for analyzing and synthesizing circuits. In programming also, we begin with the craft lore of standard algorithms and data structures before introducing any general program analysis, synthesis or verification methods.

1.2 Multiple Points of View

The range of applicability of inspection methods rests crucially on the ability to recognize familiar forms in various contexts. There are many different ways in which the recognition of familiar forms can be obscured. For example, in electrical engineering a standard circuit may not appear to be familiar because some components are in parallel rather than in series, or vice versa. Similar difficulties also arise in programs. For example, the placement of exit tests other than at the top or bottom of a loop can obscure the recognition of standard loop forms.

Various techniques have been developed in different fields to overcome such complications. These techniques are variously called equivalences, transformations, or models. All of these can be be thought of as ways of providing the user with different *points of view* on a problem. Sometimes a different point of view is necessary in order to use inspection methods at all. Sometimes several different points of view each contribute some part of the solution. For example, in the analysis and synthesis of electrical circuits, equivalence theorems (such as Thevenin-Norton) are a basic tool for rearranging the topology of circuits to match standard forms. Electrical engineers also use views in which certain features of the problem are ignored — the so-called AC (sinusoidal steady state) and DC (direct current) models are examples. In one model certain components become open circuits, while in the other they become shorts. Since the circuit in each model is simpler than in the full circuit view, the user is more likely to be able to use inspection. (It is also an important feature of these particular two views that results derived in them can be simply combined to give a complete analysis of the circuit.)

Multiple points of view are also important in understanding programs. Program transformations can be used to move the position of exit tests in loops, and thereby increase the power of inspection methods which recognize loop forms. In the area of data structures, it is often necessary to view a single structure from two different points of view, each of which captures a different generalization. For example, a Lisp list can be viewed both as a recursive structure (the tail of a list is a list) and as a labelled directed graph (where the nodes are Lisp cells connected by the CDR relation and labelled by the CAR relation). The first view is appropriate for understanding CONS and CDR as push and pop operations. The second view brings to bear a programmer's experience with standard graph manipulations in order to understand RPLACD as the operation of splicing out a node. A single Lisp list may be used in both these ways in a single program.

Another example of point of view in programming is what I call the "steady state" model of loops (and in general, recursions). In this view, exit tests are ignored in order to recognize the basic iteration and recursion forms, such as counting, summing, CAR-CDR recursion, etc. This view is similar to the AC model in electronic circuits, in that it can be simply combined with other views to construct a complete description. For example, the counting part of a loop can be abstracted as generating an infinite sequence of numbers, which is truncated by the exit test.

As we will see later in this chapter, a mechanism for representing multiple points of view is an important part of the formalization of inspection methods in programming.

Overlapping Implementations

A kind of recognition difficulty which arises often in engineering domains is when the implementations of two distinct abstract functions overlap. This means that a single component at the implementation level plays a role in two distinct forms. For example, a screw in a mechanical device may fasten two plates together and also provide a fulcrum about which to pivot a lever. In a radio-frequency amplifier, an inductor may be both part of a resonant circuit in the AC model and also part of the bias network of a transistor in the DC model. This kind of "bumming" is not just a feature of arcane programming — it is an essential part of good engineering.

For example, consider the following program which computes both the maximum and the minimum of a non-empty list of numbers.

```
(DEFUN MAX-MIN (L)
  (LET ((MAX (CAR L))
        (MIN (CAR L)))
    (MAPC '(LAMBDA (N) (COND ((> N MAX) (SETQ MAX N)))
                       (COND ((< N MIN) (SETQ MIN N))))
          (CDR L))
    (CONS MAX MIN)))
```

The standard loop plan for finding the maximum (or minimum) element of a list has three principal parts: an initialization (here (CAR L)), an enumeration of the elements of the list (here MAPC), and an accumulation which tests each element to see if it is the largest (or smallest) found so far. The diagram below indicates how MAX-MIN can be analyzed in terms of this plan.

```
        o
       / \
      /   \
     o     o
    /|\   /|\
   / | \ / | \
  o  o  o  o  o
```

The top node in this diagram represents the entire program. At the next level, the program is viewed as the combination of two plans, one which finds the maximum and one which finds the minimum. The third level shows how the more primitive components of the program are grouped and viewed as the implementation of these two plans. There are only five nodes at this level rather than six because the list enumeration is shared between the implementation of maximum and of minimum. It must be simultaneously viewed as filling a role in both plans.

This type of analysis is a violation of strictly hierarchical decomposition, which is currently the dominant technique in program design. We have found, however, that it is not always possible to maintain a strictly hierarchical analysis and at the same time capture the appropriate generalizations.

Implementation relationships are treated here as points of view which may overlap. This approach has the advantage of allowing the efficiency of implementation exemplified by the `MAX-MIN` program above (as compared to a strictly hierarchical implementation with two separate loops), while still capturing the similarities between this program and programs which calculate only the maximum or only the minimum.

1.3 The Plan Calculus

A key issue in formalizing the use of inspection methods in a particular domain is the representation of standard forms. Part of the work reported here has been to further develop a programming language independent formalism, called the plan calculus, for representing standard data and control structure forms (called *plans*) in programming.

This section introduces the plan calculus and points out some of its important features. A more detailed definition of plans is the topic of Chapter Four. The plan calculus is an outgrowth of earlier work by the author in collaboration with Shrobe [55] and Waters [56]. The important features of the plan calculus discussed in this section are as follows.

- *Wide Spectrum Specification*
- *Control and Data Abstraction*
- *Mutable Objects*
- *Programming Language Independence*
- *Multiple Points of View*
- *Additivity*
- *Verifiability*
- *Dependencies*

The plan calculus is made up of two major components: plans and overlays. Basically, a plan is the specification of a computation. Overlays represent the relationship between two different points of view on a computation, each of which is specified by a plan.

Programming is viewed here as a process involving the construction and manipulation of specifications at various levels of abstraction. In this view, there is no fundamental distinction between specifications and programs. A program (e.g. in Lisp) is merely a specification which is detailed enough to be carried out by some particular interpreter. This view is consistent with the current trend in computer science towards *wide spectrum* languages. The advantage of this approach is that various parts of a program design can be refined to different degrees without intervening shifts of formalism.

Plans

Computations are viewed here as composed of three types of primitives: operations, tests, and data objects. There are three corresponding types of primitive specifications in the plan calculus: *input-output* specifications, *test* specifications and *object type* specifications. Operations are specified by input-output specifications (preconditions and postconditions). Tests are specified by whether they succeed or fail when a given relation holds between the inputs. The primitive object types used in this work are numbers, sets and functions.

Hierarchy is represented by composite plans. Each composite plan specifies a set of local names for its parts (called *role* names) and a set of constraints which must hold between them. There are two kinds of composite plans, according to the types of the parts.

Data plans specify data structures whose parts are primitive data objects or other data structures. Data plans thus embody a kind of data abstraction. For example, List is a data plan with two roles named Head and Tail. The Head of a list may be an object of any type, but the Tail is constrained to be either an instance of List or the distinguished object, Nil ("the empty list"). Data plans are also used to represent common implementation forms. For example, a data plan called Segment is shown in Fig. 1-1. Data objects are indicated in plan diagrams by ovals. This plan has three roles named

Base (a sequence),
Upper (a natural number), and
Lower (a natural number),

and the following constraints:

(i) The Upper number is less than or equal to the length of the Base sequence.
(ii) The Lower number is less than or equal to the length of the Base sequence.
(iii) The Lower number is less than or equal to the Upper number.

This data plan (and special cases of it) is commonly used to implement other data abstractions, such as lists and queues.

Primitive data objects and data structures are *mutable*. For primitive data objects, this means that the behavior of the object can change while its identity remains the same. For example, we can specify a set addition operation in which the identical set is both the input and output. For data structures with

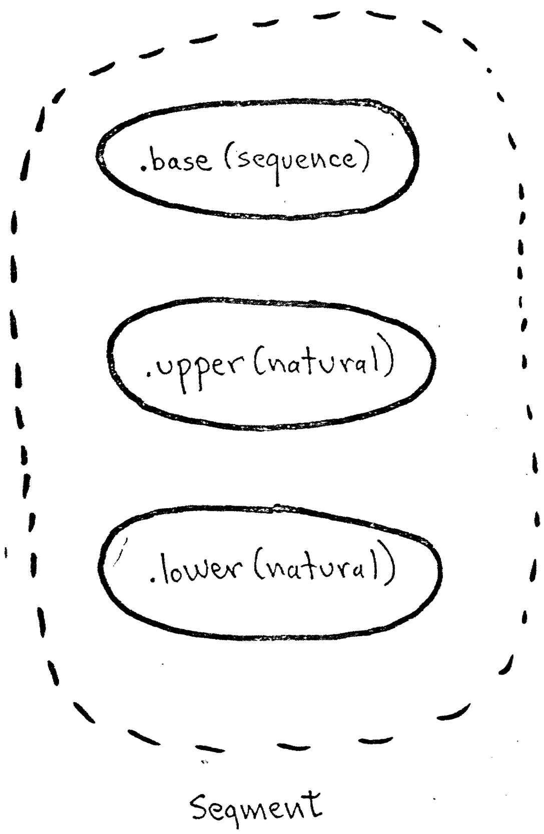

Figure 1-1. A Data Plan.

parts, such as instances of the Segment plan, mutability means that one or more of the parts may be replaced while the identity of the data structure remains the same. For example, a common operation on Segment data structures is to increment the Upper index. The semantics of mutability are part of the logical foundations of the plan calculus, which are discussed later in this section.

Temporal plans specify computations whose parts are operations, tests, data structures or other composite computations. In addition to various logical constraints between roles, such as "less than or equal", temporal plans also include *data flow* and *control flow* constraints. An example of the temporal plan for computing absolute value is shown in Fig. 1-2. Operations and tests are indicated in plan diagrams by rectangular boxes. The bottom half of test boxes are divided into cases labelled "F" for failure and "S" for succeed. This plan has three roles named

If (a test for less than zero),
Then (a negation operation), and
End (a join).[1]

Data flow constraints (solid arrows in the figure) specify correspondences between the outputs and inputs of operations and tests. Control flow constraints (hatched arrows) specify which parts of a computation are reached depending on which tests succeed or fail. Temporal plans thus embody a kind of control abstraction.

The plan calculus is to a large degree programming language independent (for a wide class of conventional sequential programming languages). This makes it possible to build a program development system which is concerned with the syntactic details of different languages only at its most superficial interface. In order to translate back and forth between a given programming language and the plan calculus, the primitives of the programming language are divided into two categories:

(i) The primitive *actions* and *tests* of the language, such as `CAR`, `CDR`, `CONS`, `NULL` and `EL` in Lisp, are represented as input-output specifications and test specifications.

(ii) The primitive *connectives*, such as `PROG`, `COND`, `SETQ`, `GO` and `RETURN` in Lisp, are represented as patterns of control and data flow constraints between operations and tests.

The translation from standard program text to an equivalent plan representation has been implemented for reasonable subsets of Lisp [53], Fortran [73] and Cobol [24]. The translation from suitably restricted plans to Lisp code has also been implemented by Waters [74].

Overlays

Overlays are the mechanism in the plan calculus for representing points of view in the programming domain. An overlay is formally a triple made up of two plans and a set of *correspondences* between roles of the two plans. Each plan represents a point of view; the correspondences express the

1. A join is a virtual entity which is needed in order to specify what the output is in each case of a conditional. Joins will be defined in Chapter Four.

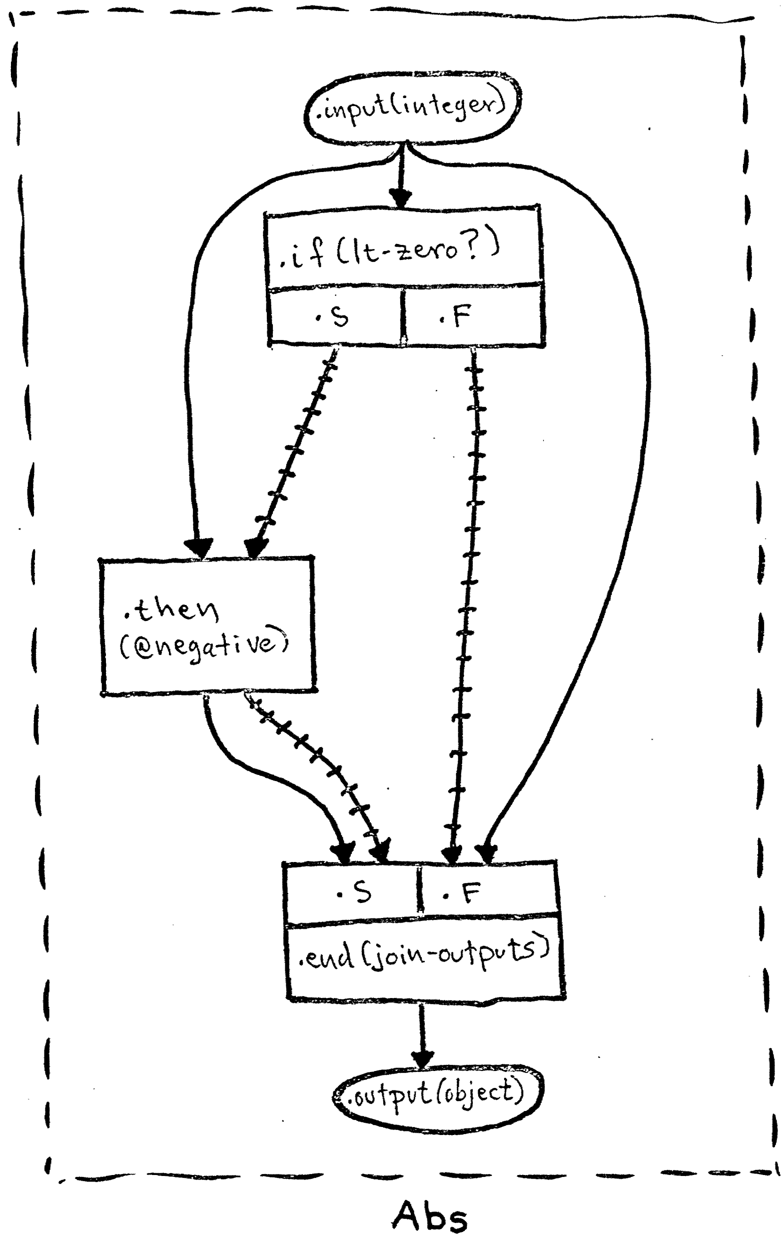

Figure 1-2. A Temporal Plan.

relationship between the points of view. Overlays are similar to Sussman's "slices", which he uses to represent equivalences in electronic circuit analysis and synthesis [69].

In addition to standard plans, there also standard overlays. For example, consider the following recursive Lisp program which copies a list.

```
(DEFINE COPYLIST
   (LAMBDA (L)
      (COND ((NULL L) NIL)
            (T (CONS (CAR L)(COPYLIST (CDR L)))))))
```

This program is an example of a singly recursive program in which there is computation "on the way up", i.e. in which the recursive invocation is not the last step in the program. Many standard recursive computations, such as list accumulation by CONSing, can be performed either "on the way down" or "on the way up." For example, the following tail recursive program, which reverses a Lisp list, performs list accumulation on the way down.

```
(DEFINE REVERSE
   (LAMBDA (L)
      (REVERSE1 L NIL))))

(DEFINE REVERSE1
   (LAMBDA (L M)
      (COND ((NULL L) M)
            (T (REVERSE1 (CDR L)(CONS (CAR L) M))))))
```

Recognition of the standard Lisp list accumulation plan in these two programs is facilitated by an overlay which expresses how, in general, to view accumulation on the way up as accumulation the way down with an intervening order reversal. This overlay is shown in Fig. 1-3. Without going into details, (For now, it is adequate just to get the idea that there are plans on both sides and correspondences between them.) consider that the plan on the left represents accumulation on the way up; the plan on the right represents accumulation on the way down. The four hooked lines between the two plans specify correspondences between the two points of view. Unlabelled correspondences (three out of the four in Fig. 1-3) are equalities. Thus the initialization of the accumulation (the Init role) is the same in both views. So are the input-output specifications of the accumulation operations (the Add role), and the final output. The most important correspondence, however, is the one labelled "reverse" in the figure. This is the correspondence which specifies that the order in which the elements of list L are accumulated in the COPYLIST program is the reverse of the order in which they are generated by the CAR-CDR part of that program. (The Lisp interpreter's stack is being used to effect the reversal.)

Notice that overlays are *symmetric*.[1] Either side can be used as a "pattern" (plans can be naturally thought of as patterns), which makes it possible to use the same overlays in both analysis and synthesis. The fact that correspondences are formally equalities means that information can propagate between points of view in both directions. For example, analysis by inspection of COPYLIST proceeds by first recognizing the standard list accumulation by CONSing plan in the point of view represented by the right

1. This is not strictly true, but only for a reason which is beyond the level of detail of this introduction.

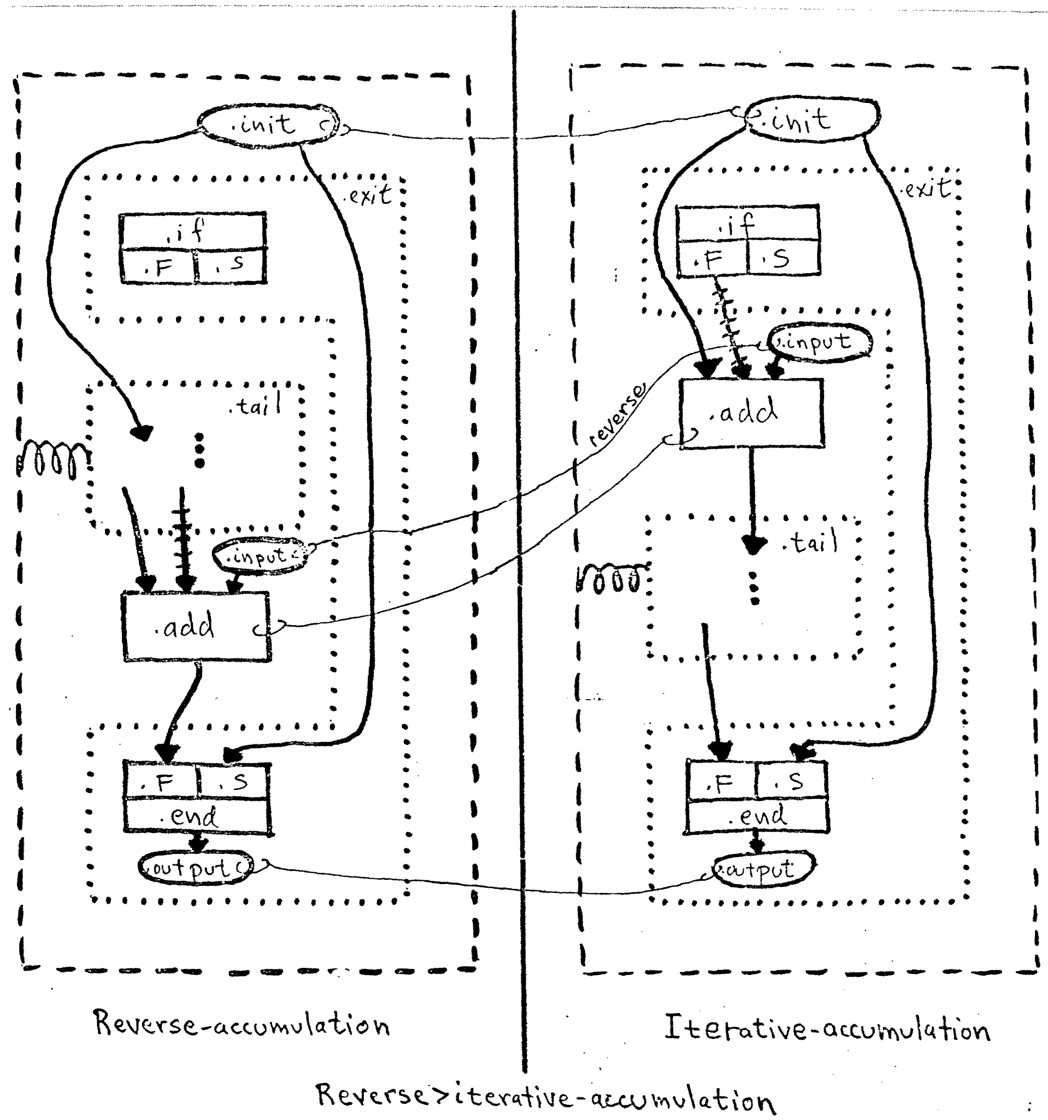

Figure 1-3. An Overlay.

hand side of the overlay in Fig. 1-3. The known properties of this plan include the fact that the final output is a list whose elements are the successive inputs to the accumulation operations, in reverse order. Propagating this information back to the original view through the correspondences and performing the algebraic simplification,

$$\text{reverse}(\text{reverse}(l)) = l,$$

leads directly to the result that the elements of the output of COPYLIST are the same as the elements of the input list, in the same order.

Implementation is also represented using overlays. One side of such an overlay is the plan representing an abstract behavior, e.g. pushing an element onto the front of a queue. The other side of the overlay is an implementation plan, e.g. storing the element in an array and adding one to an index pointer. The correspondences in such an overlay propagate information between the abstract and concrete views. Such overlays can be used both in analysis by inspection and in synthesis by inspection. In analysis by inspection, one tries to recognize known implementation plans. Once such a plan is recognized, it is replaced by (overlaid with) the corresponding abstract plan, and analysis continues similarly. Conversely, in synthesis by inspection one matches against abstract plans and instantiates implementation plans.

Logical Foundations

The remaining features of plans and overlays, namely additivity, verifiability and dependencies, all relate to the logical foundations of the plan calculus. Formally, a plan is a set of axioms in a first order logic. (The details of the axiomatization are given in Chapter Eight.) Although in fact plans are not intended to be manipulated directly as first order axioms, this logical foundation provide a semantics and a set of proof rules against which actual manipulations can be validated.

Placing plans in the paradigm of logic has several advantages. For example, *additivity* is a direct consequence of an axiomatic formalization. Combining plans has the same formal properties as the union of axiom systems, i.e. the result of combining two non-contradictory plans is always a plan which satisfies the constraints of both of the original plans. This is a desirable property not shared by other formalisms, such as program schemas. Additivity also meshes well with the principle of least commitment, which in this context means that implementation plans should have the minimum number of constraints necessary to support the implemented abstract behavior.

The logical foundations of the plan calculus are also involved in inspection methods for program verification. Verification by inspection is based on recognizing plans and applying already verified overlays. Automating the verification of overlays is not part of the research reported here. However, the logical foundations developed here do establish what needs to be proven to verify an overlay. For example, the verification of an implementation overlay entails proving that the constraints of the abstract plan are derivable from the constraints of the implementation plan together with the correspondences taken as premises.

In addition to simply recording that an overlay has been verified, it is useful to keep a record of which constraints of the implementation plan were used in the proof of which constraints of the abstract plan. This information can be extracted as a by-product of the proof process [64]. Such links are called *dependencies.* Dependencies, as part of the plan calculus, are a network of links between specifications which trace the logical derivation of one from the other. Dependencies capture a dimension of logical structure which is different from the hierarchical decomposition expressed by the roles of a plan.

Dependencies make it possible for the programmer's apprentice to explain how a program works and reason about the potential effects of a modification. For example, if you want to delete a constraint from an implementation plan, the dependencies tell you exactly which constraints of the corresponding abstract plan could become invalid. Similarly, if you change the abstract specifications of an already verified overlay, the dependencies indicate which parts of the verification need to be redone and which parts can be carried over without any extra work. The use of dependencies in reasoning about programs, especially in program evolution and modification, has been the focus of related work by Shrobe [64].

1.4 Guide to the Reader

The remaining chapters of this report can be grouped into three units. The first unit, consisting of Chapters Two, Three and Four, gives an overview of the three main areas of this work. Chapter Two is a scenario which illustrates the use of inspection methods in understanding an example program which implements a simple symbol table with hashing. Chapter Three outlines the scope of the current plan library. Chapter Four introduces the diagrammatic notation which will be used in the rest of the report to define plans and overlays.

Chapters Five, Six and Seven form a second unit, which fills in more details. Each of these chapters is an in-depth scenario of the use of inspection methods in program analysis, synthesis or verification. The example program introduced in Chapter Two is also used in each of these chapters. The style of presentation in these chapters is to introduce and explain new plans as they are needed in the example. Also, for ease of referring to previously defined plans, an index is provided at the back. If there are two page numbers listed for each item, the first is the page on which the plan or overlay diagram appears; the second is the appendix entry for that item.

The final unit, Chapters Eight, Nine and the appendix, is the most detailed and technical. Chapter Eight lays out the logical foundations of plans and overlays, including the formalization of plans involving side effects. Chapter Nine gives the detailed formalization of loop plans and temporal abstraction (a way of viewing loops in which their specifications are easily composed). These topics are treated in a more general way earlier. The appendix is a reference for the plan library, in which can be found the detailed specifications for any plans or overlays not fully described in the text.

1.5 Relation to Other Work

It is useful to distinguish three areas of concern in this work. In this section I outline some connections and comparisons with other work in these areas. The three areas are:

- *Taxonomy* - Standard programming forms and the relationships between them.

- *Formalism* - For representing programming knowledge.

- *Applications* - Analysis, synthesis, and verification of programs.

More generally, at the end of this section, I discuss related work on aspects of programming other than the use of inspection methods, such as debugging and deductive methods.

Program Taxonomies

Many people in the computer science and software engineering community have been calling for the codification of standard program forms for a long time. Two major motivations for this are: to improve software reliability and correctness, and to improve the education of programmers. For example, Dijkstra in his influential *Notes on Structured Programming* [17] called for the codification of standard program forms with associated theorems about their correctness, as follows.[1]

> "$d := D$;
> **while non** prop(d) **do** $d := f(d)$" (6)
>
> When a programmer considers a construction like (6) as obviously correct, he can do so because he is familiar with the construction. I prefer to regard his behavior as an unconscious appeal to a theorem he *knows*, although perhaps he has never bothered to formulate it; and once in his life he has convinced himself of its truth, although he has probably forgotten in which way he did it and although the way was (probably) unfit for print. But we could call our assertions about program (6), say, "The Linear Search Theorem" and knowing such a name it is much easier (and more natural) to appeal to consciously.
>
> ...it might be a useful activity to look for a body of theorems pertinent to such programs.

More recently, Floyd in his 1978 ACM Turing Award Lecture [27] spoke as follows about the importance of teaching the standard forms of programming to new programmers, as compared with emphasizing the primitive programming language constructs. (Floyd calls these forms *paradigms* and is particularly interested in very general ones, such as "divide and conquer").[2]

> To the teacher of programming, even more, I say: identify the paradigms *you* use, as fully as you can, then teach them explicitly. They will serve your students when Fortran has replaced Latin and Sanskrit as the archetypal dead language.

1. p. 10.
2. p. 459.

Many people have answered these calls, using a variety of expressive tools and covering a range of programming areas. I group these efforts roughly into two categories.

In the first category are those who have tried to give wide coverage of the basic forms of everyday programming, such as the standard manipulations involving of sets, directed graphs and linear data structures (lists and sequences). Most prominent in this category is the work of Knuth [37]. In three volumes, Knuth uses a mixture of mathematics, example programs and expository English text to communicate his "programmer's craft" in fundamental algorithms (manipulations on linear lists and trees), semi-numerical algorithms (random numbers and arithmetic), sorting and searching. There are also many one-volume text books [1] which have a similar format, but are less comprehensive.

In the second category, I put those whose have focused on a more particular programming domain. Not surprisingly, work in this category is also characterized by more formal representations (some of which will be discussed in the next section). Domains that have been studied in some depth include algorithms on sequences [50,52], sorting [32], standard loop forms [49,73], set implementations [61,57], and the implementation of associative data structures [58].

This work falls partly in both categories. The contents of the current plan library is mostly the result of generalizing the plans required for an in-depth understanding of a particular example program — the implementation of a symbol table using hashing, which is introduced in the scenario in Chapter Two. This example program was chosen because it involves many different techniques which are representative of the domain of routine symbolic manipulations (sets, lists, etc.). I believe that a library which is adequate for this example is a good start towards complete coverage of the domain. The small fraction of plans in the current library which are not directly motivated by the symbol table example fall into two categories. Some of these are obviously important basic plans which don't happen to be used in the example, such as counting and accumulation loops. Other plans are included to fill gaps in the taxonomic structure of the library, such as the plan for splicing into a list (only splicing out appears in this particular symbol table). Barstow's work [6] is similar in depth and breadth.

Other Formalisms

Past efforts to construct knowledge bases for automatic or partially automated programming have used the following formalisms: program schemas [29], program transformations [15,5,12], program refinement rules [6], and formal grammars [59]. Although each of these representations has been found useful in certain applications, none combines all of the important features of the plan calculus listed above.

For example, program schemas (incomplete program texts with constraints on the unfilled parts) have been used by Wirth [76] to catalog programs based on recurrence relations, by Basu and Misra [7] to represent typical loops for which the loop invariant is already known, and by Gerhart [29] and Misra [50] to represent and prove the properties of various other common forms. Unfortunately, the syntax of conventional programming languages is not well suited for the kind of generalization needed in this endeavor. For example, the idea of a search loop (a standard programming form) expressed informally in English should be something like the following.

> A *search loop* is a loop with two exits in which a given predicate (the same one each time) is applied to a succession of objects until either the predicate is satisfied, in which case that object is made available for use outside the loop, or the objects to be searched are exhausted.

In Lisp, as in other languages, this kind of loop can be written in innumerable forms, many of which are syntactically (and structurally) very different, such as:

```
(PROG ()
   LP (COND (exhausted (RETURN NIL)))
      ...
      (COND ((predicate current)(RETURN current)))
      ...
      (GO LP))
```

or with only one RETURN instead of two,

```
(PROG ()
   LP (COND (exhausted NIL)
            (T ...
               (COND ((predicate current)
                      (RETURN current)))
               ...
               (GO LP))))
```

or even recursively, e.g.

```
(DEFINE SEARCH ()
   (COND (exhausted NIL)
         (T ...
            (COND ((predicate current) current)
                  (T ...
                     (SEARCH))))))
```

The problem here is that conventional programming languages are oriented towards specifying computations in enough detail so that a simple local interpreter can carry them out. Unfortunately a lot of this detail is often arbitrary and conceptually unimportant. In the plan calculus, all three of the schemas above (and many other such variations) are expressed by a single plan.

A new generation of programming languages descended from Simula [16], such as CLU [38] and Alphard [77], provide a syntax for specifying standard forms such as the search loop in a more canonical way. However, there are two more fundamental difficulties with using program schemas to represent standard program forms, which Simula and its descendants do not solve. First, programs (and therefore program schemas) are not in general easy to combine, nor are they additive. This means that when you combine two program schemas, the resulting schema is not guaranteed to satisfy the constraints of both of the original schemas, due to such factors as destructive interactions between variable assignments. Second, existing programming languages do not allow multiple views of the same program or overlapping module hierarchies. I believe the reason for this is that a program is still basically thought of, from the standpoint of these languages, as a set of instructions to be executed, rather than as a set of descriptions (e.g. blueprints) which together specify a computation.

Another commonly used formalism for representing abstract programming forms is *flowchart schemas*. Originally developed by Ianov in 1960 [36], flowchart schemas are a network-like connection of test and operation boxes. This formalism has the features of being programming language independent and having logical foundations. (Manna gives an excellent tutorial on the formalization and use of flowchart schemas in his book on the mathematical theory of computation [40].) Flowchart schemas capture control flow abstraction in a very natural and intuitive way. However, the only method they provide for expressing the flow of data between operations is variable assignment. Unfortunately, the use of variables in this way destroys additivity the same as for programming languages.

This problem with flowchart schemas can be fixed by combining flowchart schemas with another network-like formalism, the *data flow schemas* of Dennis [19]. In data flow schemas, operations have local port names and data flow is represented by port-to-port connections. The synthesis of these two types of schemas is essentially the temporal plan formalism used here. Temporal plans, however, have the additional feature that mutable objects are representable, which is not the case in data flow schemas.

A currently popular approach for specifying data abstractions is the *algebraic axiom* formalism [33,39,30]. Though data plans are formally equivalent to abstract data types, in practice the approach in this work is somewhat different (mostly due to concern with mutable objects). In the algebraic axiom framework, there are no mutable objects or side effects. For example, in the standard algebraic axiomatization of stacks one defines the following three primitive functions on stacks[1]

push: stack × object → stack
pop: stack → stack
top: stack → object

and the following set of algebraic equations.

$$\mathrm{top}(\mathrm{push}(x,y))=y$$
$$\mathrm{pop}(\mathrm{push}(x,y))=x$$

In this work, however, similar behavior is formalized differently. The only primitive functions on a data structure are its roles, which are thought of as access functions. For example, the fundamental singly recursive data structure is called List. The two primitive access functions on lists are[2]

head: list → object
tail: list → list

In this framework, operations such as Push, Pop, and Top, are non-primitive concepts which are specified by input-output specifications roughly as follows.

(i) A **Push** operation take as input a list and an object; its output is a list whose head is the input object and whose tail is the input list.

1. We do not worry about the empty stack in this example.
2. Again we do not worry about the empty case, since it is not relevant to the comparison being made in this section. The formalization of data plans is presented more completely in Chapter Eight.

(ii) A **Pop** operation takes as input a list; its output is the tail of the input list.
(iii) A **Top** operation takes as input a list; its output is the head of the input list.

Side effects are specified in this framework by specifying an operation in which the same object is both input and output, but in which parts of that object (i.e the values of primitive access functions) are different before and after. Recently, Guttag and Horning [34] have taken a similar approach. They call the part of their system in which side effects are specified "routines" and use the predicate transformer notation instead of preconditions and postconditions.

Other work on representing mutable data objects and side effects includes Early [23], Burstall [11] and Yonezawa [78]. Of these, the V-graphs of Early are the most similar to data plans. Early also takes access paths as the only primitive functions, and specifies side effect operations as transformations on the part structure of data objects.

Currently the most common way to represent relationships between standard forms (typically implementation/abstraction relationships) is via program transformations or program refinement rules [6]. As compared to overlays, these formalisms have two serious problems which stem from their lack of neutralness between analysis and synthesis. An overlay in the plan calculus, as in Fig. 1-4, is made up of two plans and a set of *correspondences* between the parts of the two plans. Each plan represents a point of view; the correspondences express the relationship between the points of view. For example, in an implementation overlay the plan on the right hand side is the abstract description and the plan on the left hand side is an implementation. It is important, however, that either plan can be used as the "pattern". In a typical program synthesis step using overlays the right hand plan is used as the pattern and the left hand plan is instantiated as a further implementation. Conversely, in a typical analysis step, the left hand plan serves as the pattern and the right hand plan is instantiated as a more abstract description. With both program refinement rules and knowledge-based[1] program transformations this sort of symmetric use is not possible since the right hand side of a transformation or refinement rule is typically a sequence of substitutions or modifications to be performed, rather than a pattern.

A second problem stemming from the asymmetry of program transformations and refinement rules is their lack of verifiability. The correctness of an overlay in the plan calculus is verified by proving essentially that the constraints of the plan on the left hand side, together with the correspondences (which are formally a set of equalities between terms on the left and terms on the right) imply the constraints of the plan on the right hand side. Neither Balzer's transformation language nor Green and Barstow's refinement tree notation has been adequately formalized to permit the question of correctness to be addressed. The recent work of Broy and Pepper [10] is an improvement in this direction, since their transformations have program forms on both the left and right hand sides, with associated proof rules. Unfortunately, they use program schemas as the representation of the standard forms which has the difficulties discussed above.

---

1. As opposed to the folding-unfolding and similar transformations of Burstall and Darlington [12] which are intended to be a small set of very general transformations which are formally adequate, but which must be composed appropriately to construct intuitively meaningful implementation steps.

Another formalism some have found attractive for codifying programming knowledge is formal grammars. For example, Ruth [59] constructed a grammar (with global switches to control conditional expansions) which represented the class of programs expected to be handed as exercises in an introductory PL/1 programming class. This grammar was used in a combination of top-down, bottom-up and heuristic parsing techniques in order to recognize correct and near-correct programs. Miller and Goldstein [47] also used a grammar formalism (implemented as an augmented transition network) to represent classes of programs in a domain of graphical programming with stick figures. The major shortcoming of these grammars from the point of view of the programmer's apprentice is their lack of a clear semantics upon which a verification methodology can be based.

Computer Aided Program Development Systems

The application area to which this work is aimed can be generally described as computer aided environments for program development. In particular, this work is part of a project [56] aimed at developing what we call a *programmer's apprentice* system. What distinguishes a programmer's apprentice from existing systems is the level of program understanding shared between the user and the system.

Existing program development systems provide various types of services at different levels of understanding. The level of least understanding is when the system manipulates everything as text strings. At this level, various kinds of useful bookkeeping can be provided, such as keeping track of versions of source code, test data and documentation [2,22].

The next level of understanding is when the system is able to parse the syntax of the user's programming language. At this level it is possible to provide many more useful services, such as structure editors [20] and cross-referencing [70]. If in addition the system can interpret the semantics of the programming language, then further analysis and verification assistance is possible, such as symbolic interpreters [13,3] and verification condition generators [51]. A slight step above the programming language understanding level are systems which support the syntax of a more abstract design formalism [75].

I believe that current systems are quickly approaching fundamental limitations to the services they can provide due to fact that they understand programs only at the level of the programming language. I believe the next major step, represented by the programmer's apprentice, is to understanding based on a library of standard programming forms. This will make it possible for the system to apply inspection methods to the analysis, synthesis and verification of programs. The scenario in the next chapter elaborates what a programmer's apprentice could do.

Other Aspects Of Programming

Inspection methods are certainly not the whole story in programming. Programmers are not always faced with totally familiar problems. Miller [48] has studied and catalogued some very general problem decomposition methods which programmers can apply when faced with unfamiliar problems.

Sussman [67] has explored the role of debugging when plans are "almost right". Finally, Manna and Waldinger [41] have explored the applicability of deductive methods to programming.

Other Engineering Problem Solving

The study of problem solving in other areas of engineering has had a strong influence on this work. In particular, the notion of the plan for a program is similar to the plans for electrical circuits in the work of Brown [9] and de Kleer [18]. Freiling [28] also used a similar approach in the area of mechanical engineering.

CHAPTER TWO

PROGRAMMER'S APPRENTICE SCENARIO

2.1 Introduction

A library of plans opens up many new possibilities for what a computer aided program development system can do to help a programmer. This chapter illustrates some of these new possibilities, without going into too much detail. Chapters Five, Six and Seven go into more depth on how the behavior illustrated here can be implemented.

Many different activities are interwoven in the programming process. These activities can be roughly dividing into three major areas: analysis, synthesis and verification. Analysis activities in general involve determining properties of a program which are not explicit in its definition (usually by decomposing it into parts). Synthesis in general involves refining an abstract description into one which is more detailed in the appropriate sense for some target machine. Verification in general has to do with detecting errors and constructing arguments as to why a program works.

A program development system can aid a programmer in all three of these areas. For a programmer's apprentice system, in particular, this means the same library of plans is used for analysis, synthesis and verification by inspection. For example, suppose there is a plan which captures the idea of iteration with a "trailing" value, as illustrated by the following code.

```
(PROG (CURRENT PREVIOUS)
      ...
   LP (SETQ CURRENT ...)
      ...
      (SETQ PREVIOUS CURRENT)
      (GO LP))
```

If this plan is in the library, the system should be able to recognize its use in programs it hasn't seen before; it should be able to synthesize programs using this plan; and it should be able to detect errors in the use of this plan, such as incorrect initialization. This factorization of knowledge is an important feature of the design of programmer's apprentice.

The scenario in this chapter portrays a system in which inspection methods for program analysis, synthesis and verification are fully integrated. At the time of this writing, an integrated system with these capabilities has not yet been implemented. However, several of the major functions portrayed in the scenario have been implemented separately in experimental form. Waters has implemented a system which translates Lisp code to the plan calculus and performs some further analysis on the resulting plans. Shrobe has implemented a system which verifies plans by symbolic evaluation. Although a complete synthesis system has yet been implemented, Waters has implemented the bottom-end module for this which translates suitably detailed plans to Lisp code. Finally, an experimental program for automatically drawing plan diagrams from the system's internal representation has also been implemented.

What we will see in this scenario is basically the top-down, stepwise refinement style of program development. This should not be taken as an endorsement of this methodology to the exclusion of all others. I believe that the appropriate style of development depends greatly on the particular programming task. A strongly top-down development, such as in this scenario, is appropriate when the programming task is very familiar. On other occasions, a bottom-up or middle-out methodology may be more appropriate. This research is aimed at developing the representations and basic capabilities which are needed to support program development in any order.

2.2 The Scenario

In this scenario, we see an expert user interacting with a programmer's apprentice system to develop the implementation of a symbol table using hashing. Lines typed by the user are in lower case; lines typed by the system are in upper case. The user communicates with the system in a simple English-like command language. The important part of this language is not its syntax, but the *vocabulary* of programming terms shared between the user and the system.

```
> let a "symbol table" be a finite set.
OK
```

The system initially doesn't know anything about symbol tables *per se.* The user thus begins by introducing "symbol table" as a new term (indicated by quotation marks). The term finite set is known to the system.

```
> implement a symbol table.

  APPLICABLE IMPLEMENTATIONS FOR: SYMBOL-TABLE (A FINITE-SET)
        1. LIST
        2. SEQUENCE
        3. LABELLED-THREAD
        4. DISCRIMINATION
  ? )
```

Here the system presents a *menu* of ways it knows for implementing a finite set. A completely menu-driven scenario is shown in Chapter Six. Such menus are computed dynamically by searching the current library for overlays between finite sets and other data structures. At present there are only four choices; more implementations will be added to the library in the future. The user declines to choose an option, asking for some documentation first.

```
> describe a discrimination function.

A DISCRIMINATION FUNCTION IS A FUNCTION FROM OBJECTS TO SETS
SUCH THAT, IF AN OBJECTS IS IN A SET IN THE RANGE OF THE FUNCTION,
THEN THE FUNCTION APPLIED TO THE OBJECT EQUALS THE SET.

> implement a symbol table as a discrimination function.
OK
```

A discrimination function is a function which partitions its domain into a disjoint set of sets (called buckets). Discrimination functions are part of the system's library of standard forms. The formal definition of discrimination function presented above is not a "canned" text string, but is constructed from the logical formula which defines discrimination function in the system's library.[1] The user's first refinement step is then to implement the symbol table as a discrimination function.

The library also contains plans and overlays involving discrimination functions. The overlay invoked by the user above says that a discrimination function can be viewed as implementing a set which is the union of its buckets.

```
> show code for addition to a symbol table by side effect.

        (DEFINE SYMBOL-TABLE-ADD
           (LAMBDA (OLD INPUT)          ;MODIFIES OLD.
              (PROG (BUCKET)
                    (SETQ BUCKET (APPLY OLD INPUT))
                    (UPDATE-FUNCTION OLD
                                     BUCKET
                                     (BUCKET-ADD BUCKET INPUT)))))

        (DEFINE BUCKET-ADD
           (LAMBDA (OLD INPUT)
              ...))

        (DEFINE UPDATE-FUNCTION
           (LAMBDA (OLD VALUE INPUT)    ;MODIFIES OLD.
              ...))
```

The system knows how to implement addition to a set implemented as a discrimination function. One way of displaying this knowledge is for the system to generate code from its current plan representation of the design thus far. As can be seen above, this code has gaps in it (indicated by three dots). These gaps correspond to the fact that it doesn't yet know how the discrimination function or the buckets are implemented.

The arguments to the procedure SYMBOL-TABLE-ADD are the symbol table to be modified (OLD) and the entry to be added (INPUT). At this point in the design, all the system knows is that the old symbol table is a function, but not how that function is implemented. The discrimination function may later in the design turn out to be implemented by a Lisp procedure, by a data structure, or by a combination of the two. The code (APPLY OLD INPUT) in the first line of SYMBOL-TABLE-ADD is therefore used by the system to express the general idea of applying the discrimination function to get the appropriate bucket in which to add the given entry. Similarly, the procedure UPDATE-FUNCTION is a placeholder for the operation of modifying an OLD function such that all domain elements which used to map to a given range VALUE (in this case the bucket before addition), will afterwards map to the INPUT (in this case the bucket after addition)..

1. See the appendix for the definition in logical form. The technology for making this kind of a straightforward translation exists [46].

Despite the arguments made earlier as to why code is not an appropriate internal representation for the library of standard forms, code can still be a convenient language for communicating between the system and the user in certain circumstances. Also code has the advantage of being more familiar to the user than the plan calculus. Notice also that the system has annotated the code it has written with comments. We will see later in the scenario that the system also reads the comments on any code that the user types in.

```
> describe symbol-table-add.

SET ADDITION TO A DISCRIMINATION FUNCTION IS IMPLEMENTED BY THREE STEPS --
        DISCRIMINATE: APPLY THE DISCRIMINATION FUNCTION,
        ACTION: SET ADDITION,
        UPDATE: MODIFY THE DISCRIMINATION FUNCTION WITH A NEW VALUE,
SUCH THAT
  THE OUTPUT OF THE DISCRIMINATE STEP IS THE OLD SET OF THE ACTION STEP;
  THE INPUT OBJECT OF THE ACTION IS THE INPUT OBJECT OF THE DISCRIMINATE STEP;
  THE OUTPUT OF THE DISCRIMINATE STEP IS THE VALUE INPUT TO THE UPDATE STEP;
  THE NEW SET OF THE ACTION STEP IS THE INPUT OBJECT TO THE UPDATE STEP;
  AND THE DISCRIMINATION FUNCTION IS THE OLD FUNCTION INPUT TO THE UPDATE STEP.
```

In response to a request from the user, the system here gives a toplevel logical decomposition of the code it has written. This description is again not "canned" text, but can be generated from the plan representation as needed. For this sort of information, displaying the plan diagram shown in Fig. 2-1 is probably superior.

```
> implement the discrimination function as a keyed discrimination.
OK

> describe keyed discrimination.

A KEYED DISCRIMINATION IS A COMPOSITION OF TWO FUNCTIONS --
        ONE: A FUNCTION ("THE KEY FUNCTION"),
        TWO: A FUNCTION ("THE BUCKET FUNCTION"),
SUCH THAT THE COMPOSITION, VIEWED AS A FUNCTION, IS A DISCRIMINATION
FUNCTION.
```

The design of the symbol table continues in small steps. The next step here is to decompose the discrimination function into two functions: one which maps from objects to keys; and one that maps from keys to buckets. The strings in quotation marks above are "canned" text which is attached to roles of the plan to give better words than "the one function" and "the two function", which would be generated automatically.

The system knows quite a bit about functional compositions. For example, it knows that the range of the first function must be a subset of the domain of the second function. It also knows that to update a function implemented as the composition of two functions, it suffices to update the second function. Both of these pieces of information will be used later in the scenario.

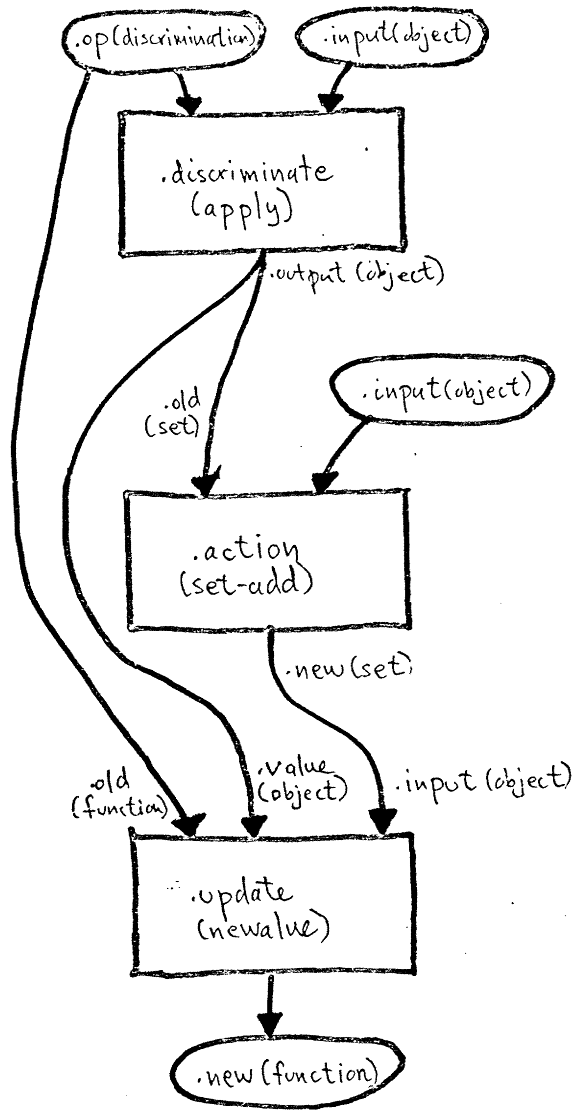

Figure 2-1. Discriminate, Action and Update Plan for Addition to Symbol Table.

```
> the key function of the keyed discrimination is car.
OK

> implement the bucket function of the keyed discrimination as a hashing.
OK

> describe hashing.

A HASHING IS A COMPOSITION OF TWO FUNCTIONS --
        ONE: A FUNCTION ("THE HASH FUNCTION"),
        TWO: AN IRREDUNDANT SEQUENCE ("THE TABLE") .
```

The final step in the refinement of the symbol table data structure is to introduce hashing. The basic idea of hashing is to decompose a function (in this case the function from keys to buckets) into two functions: a many-to-one function which maps from the domain of the original function to an initial interval of the natural numbers; and a one-to-one function which maps from the natural numbers to the range of the original function. (Sequences are formalized in this system as functions on initial intervals of the natural numbers. An irredundant sequence is one in which no two terms are equal.)

```
> show symbol-table-add.

        (DEFINE SYMBOL-TABLE-ADD
           (LAMBDA (TABLE INPUT)                ;MODIFIES TABLE.
              (PROG (INDEX)
                    (SETQ INDEX (HASH (CAR INPUT)))
                    (ARRAYSTORE TABLE INDEX
                                (BUCKET-ADD (ARRAYFETCH TABLE INDEX)
                                            INPUT)))))

        (DEFINE HASH
           (LAMBDA (INPUT)
              ...))

        (DEFINE BUCKET-ADD
           (LAMBDA (OLD INPUT)
              ...))
```

The user has asked the system to redisplay code for SYMBOL-TABLE-ADD based on the additional specifications provided thus far. Notice that the system still doesn't know how the buckets are implemented, so BUCKET-ADD is still just a placeholder. However, the system does now have enough information to make some other implementation decisions. For example, since the hashing function does not need to be modified, it has been implemented as a Lisp procedure.[1] The only function that needs to be modified now is the table (a sequence). The system has also taken the initiative of implementing this sequence as a Lisp vector. Modifying a Lisp vector is achieved by ARRAYSTORE.

If desired, the user could have been queried about each of these decisions. However, it is probably preferable to have the system take the initiative at this low level and make it possible for the user to

1. There is nothing in the current plan library concerning what makes a good hashing function. This is in the domain of numerical computation and theory, which is outside of the focus of this work.

retract decisions later.[2]

```
> show code for associative retrieval from a symbol table.

        (DEFINE SYMBOL-TABLE-RETRIEVE
           (LAMBDA (TABLE INPUT)
              (BUCKET-RETRIEVE (ARRAYFETCH TABLE (HASH INPUT))
                               INPUT)))

        (DEFINE BUCKET-RETRIEVE
           (LAMBDA (BUCKET INPUT)
              (PROG (OUTPUT)            ;SEARCH LOOP
                    (COND (... (RETURN NIL)))
                    (SETQ OUTPUT ...)
                    (COND ((EQ (CAR OUTPUT) INPUT)
                           (RETURN OUTPUT)))
                    (GO LP)))))
```

Associative retrieval is a standard specification known to the system. Conceptually, it has three inputs: a set, a key function, and a search key. It also has two cases: if there is a member of the set such that the key function applied to it equals the search key, then the retrieval succeeds and its output is such a member; otherwise, it fails.[3]

As can be seen from the code above, the system also knows the standard plan for implementing associative retrieval from a set implemented as a keyed discrimination, namely: apply the bucket function to the search key to obtain a bucket; and then perform associative retrieval from the bucket using the same key function and search key. Notice that the respective sets (either the whole table or the bucket) and the search key (INPUT) are the formal parameters of SYMBOL-TABLE-RETRIEVE and BUCKET-RETRIEVE in the code above, while the key function, CAR, is coded in line. This coding does not cause any loss of modularity, since the purpose of that particular use of CAR is preserved in the plan representation.

The gaps in BUCKET-RETRIEVE are due to the fact that the user has not yet specified how the buckets are implemented. However, the systems does know that this procedure conceptually has two cases. Procedures in Lisp can have only one return point. The system has thus decided to use the standard technique of a *flag* to encode two cases — a return value of NIL signals the failure case. Minor programming techniques, such as the use of flags, can also be captured in the plan representation.

The following illustrates another kind of interaction between the user and the system. In addition to being able to read each other's code, the user and system can also *edit* each other's code. This interaction also shows how the plan library can provide the user with a sort of "macro expansion" facility for writing code more easily and correctly. For example, the user below invokes the standard plan for generating successive elements of a Lisp list by CAR and CDR, truncated by NULL. The code that is generated for this plan in this context is underlined below.

2. A truth maintenance system [21,43] can facilitate this kind of retraction. The issues involved in incremental modification are an important part of the programmer's apprentice project in general, but are outside of the focus of this work.
3. This way of specifying associative retrieval generalizes well to multiple key retrieval.

```
> insert code for truncated list generation of bucket in bucket-retrieve.

        (DEFINE BUCKET-RETRIEVE
           (LAMBDA (BUCKET INPUT)
              (PROG (OUTPUT)     ;LIST GENERATION AND SEARCH LOOP
                 LP (COND ((NULL BUCKET)(RETURN NIL)))
                    (SETQ OUTPUT (CAR BUCKET))
                    (COND ((EQ (CAR OUTPUT) INPUT)
                           (RETURN OUTPUT)))
                    (SETQ BUCKET (CDR BUCKET))
                    (GO LP)))))
```

The important difference between what takes place here and a conventional macro facility is that here the insertion of new material takes place in the plan calculus, where the benefits of additivity hold. Only afterwards is code regenerated from the augmented plan.

Notice that the user here has indirectly told the system that the buckets of the symbol table are implemented as lists. This gives the system enough information to complete the implementation of BUCKET-RETRIEVE by identifying the exit test of the list generation with the failure case of the associative retrieval, (RETURN NIL), and by setting OUTPUT to the successively generated list elements.

```
> show symbol-table-add.

        (DEFINE SYMBOL-TABLE-ADD
           (LAMBDA (TABLE INPUT)                ;MODIFIES TABLE.
              (PROG (INDEX)
                    (SETQ INDEX (HASH (CAR INPUT)))
                    (ARRAYSTORE TABLE INDEX
                                (CONS (ARRAYFETCH TABLE INDEX)
                                      INPUT)))))
```

Given that buckets are implemented as lists, the system assumes the most straightforward implementation for BUCKET-ADD which is simply CONS, as shown above.

```
> describe the implementation of a symbol table.

A SYMBOL TABLE IS IMPLEMENTED AS A DISCRIMINATION FUNCTION.

THE BUCKETS OF THE DISCRIMINATION FUNCTION ARE IMPLEMENTED AS LISTS.

THE DISCRIMINATION FUNCTION IS IMPLEMENTED AS A KEYED DISCRIMINATION.

THE KEY FUNCTION OF THE KEYED DISCRIMINATION IS CAR.
THE BUCKET FUNCTION OF THE KEYED DISCRIMINATION IS IMPLEMENTED AS A HASHING.

THE HASH FUNCTION OF THE HASHING IS IMPLEMENTED AS A LISP PROCEDURE.
THE TABLE OF THE HASHING IS IMPLEMENTED AS A LISP VECTOR.
```

Above we see how the programmer's apprentice can provide an interactive documentation facility. This can be particularly useful if more than one programmer is working on a program, or if a program needs to be modified a long time after it was written.

```
> show code for associative deletion from a symbol table by side effect.

        (DEFINE SYMBOL-TABLE-DELETE
           (LAMBDA (TABLE INPUT)            ;MODIFIES TABLE.
              (PROG (INDEX)
                    (SETQ INDEX (HASH INPUT))
                    (ARRAYSTORE TABLE INDEX
                            (BUCKET-DELETE (ARRAYFETCH TABLE INDEX))))))

        (DEFINE BUCKET-DELETE
           (LAMBDA (BUCKET INPUT)
              (COND ((NULL BUCKET) NIL)
                    ((EQ (CAAR BUCKET) INPUT)
                     (BUCKET-DELETE (CDR BUCKET) INPUT))
                    (T (CONS (CAR BUCKET)
                             (BUCKET-DELETE (CDR BUCKET) INPUT))))))
```

Associative deletion is also a standard specification known to the system. Like associative retrieval, it has three inputs: a set, a key function, and a key. Its output is the input set minus those members for which the key function applied to them equals the input key. The implementation of associative deletion from a set implemented as a discrimination function is a similar three step plan to the set addition plan introduced earlier, namely: apply the discrimination function to get a bucket, perform the same associative deletion on the bucket to get a new bucket, and then modify the discrimination function to incorporate the new bucket. The system has generated code for this plan as shown above.

Notice that associative deletion from the bucket list is implemented by the system in the straightforward manner which copies the list. In the next frame, we will see that the user has something more clever in mind, and therefore intervenes to provide his own more efficient code for deleting from the bucket by side effect.

```
> edit bucket-delete

        (define bucket-delete
           (lambda (bucket input)              ;modifies bucket.
              (prog (p q)
                    (setq p bucket)
                 lp (cond ((eq (caar p) input)
                           (rplacd q p)        ;splice out.
                           (return bucket)))
                    (setq q p)
                    (setq p (cdr p))
                    (go lp))))

WARNING! THE LOOP IN BUCKET-DELETE IS ALMOST A
TRAILING GENERATION AND SEARCH,
        CURRENT: P
        PREVIOUS: Q
        EXIT: (COND ((EQ (CAAR P) ...)))
        ACTION: (CDR P)
EXCEPT THAT THE OUTPUT OF THE ACTION IS NOT EQUAL TO THE
INPUT OF THE EXIT TEST.
```

Here we see an example of inspection methods used for verification. The user has attempted to code a generation and search loop with a trailing value and has not gotten it quite right.[1] The plan in the library for trailing generation and search has the roles for the current value, the previous value, the exit test, and the generating action on each iteration, with roughly the following constraints between them:

(i) The output of the action is equal to the input of the action on the next iteration.
(ii) The output of the action is equal to the input of the exit test.
(iii) The current value is equal to the input of the exit test.
(iv) The current value is equal to the previous value on the next iteration.
(v) The current value and previous value are outputs of the loop.

In a near-miss recognition, most but not all of the constraints of a plan are satisfied. In this example, constraint (ii) is not satisfied as indicated by the system in the warning message above.[2] The details of how this recognition takes place are explained in Chapter Seven.

Verification by inspection yields a much more meaningful diagnostic than would be given by other methods of detecting this error. For example, running the code above with certain inputs would result in the Lisp interpreter halting at the RPLACD with an error message such as the following.

```
; NIL BAD ARG - RPLACD
```

In general, correcting errors is more difficult than detecting them. For example, it is hard for the system to know whether a near-miss is actually an error or just a new variation on a plan it doesn't know about. The programmer's apprentice will thus in general rely on the user to correct errors. The user's response to the warning message above is shown underlined below.

```
> edit bucket-delete

        (define bucket-delete
           (lambda (bucket input)             ;modifies bucket.
              (prog (p q)
                    (setq q bucket)
                 lp (setq p (cdr q))
                    (cond ((eq (caar p) input)
                           (rplacd q p)       ;splice out.
                           (return bucket)))
                    (setq q p)
                    (go lp))))

WARNING! THE CODE (RPLACD Q P) IS NOT RECOGNIZED AS SPLICEOUT.
SUGGEST (RPLACD Q (CDR P)) ? yes
```

Unfortunately, there is yet another error in the user's code. The user has asserted in a comment above that the line with RPLACD implements splicing an element out of the list. However, the system finds that this code does not match its library plan for splicing out. In this case, since the user has stated his intention in a comment, the system can go as far as offering a correction, which the user accepts.

1. Whether or not the user should have known that this particular plan was in the library is irrelevant. The point here is to illustrate the idea of a program understanding system that "looks over the shoulder" of the user.

2. This does not, however, mean the user's code won't work; only that the system does not have a theory by which it can understand what the code does.

Notice that at different points in this scenario, Lisp lists have been viewed both recursively and as directed graphs. The first version of BUCKET-DELETE was a standard recursion on the tail of the list; while the version above is in the domain of graph manipulations involving successor nodes and modifying arcs between them.

Now that BUCKET-DELETE has been re-implemented using side effects, a more efficient implementation of SYMBOL-TABLE-DELETE is possible, in which the ARRAYSTORE is necessary only if the first element of the bucket is deleted.

```
> show symbol-table-delete.

        (DEFINE SYMBOL-TABLE-DELETE
           (LAMBDA (TABLE INPUT)        ;MODIFIES TABLE.
              (PROG (INDEX BUCKET)
                    (SETQ INDEX (HASH INPUT))
                    (SETQ BUCKET (ARRAYFETCH TABLE INDEX))
                    (COND ((EQ (CAAR BUCKET) INPUT)
                           (ARRAYSTORE TABLE INDEX (CDR BUCKET)))
                          (T (BUCKET-DELETE BUCKET))))))
```

To come to this implementation, the system has done some analysis of side effects by inspection. Specifically, there are plans and an overlay in the library which say that one way to modify a function (change the associations between domain and range elements by adding a new range element) is to modify an old range element. Applied to this program, this overlay allows the system to view the deletion of an element from the bucket by side effect as the implementation of the modification of the discrimination function.

Analysis by inspection is also in operation here. By recognizing the user's BUCKET-DELETE code as a trailing generation and search plan, the system derives some important additional properties of this procedure. In particular, it knows that this procedure only searches internal nodes of the bucket list, and that it only finds the first node which has the given key. With regard to the first property, there is a plan in the library which combines an internal deletion with a conditional test on the first node to achieve a complete deletion. The system has used this plan to arrive at the code above. The second property is propagated up to the specifications of SYMBOL-TABLE-DELETE, as shown below.

```
> describe preconditions of symbol-table-delete.

THERE EXISTS A UNIQUE "X" SUCH THAT X BELONGS TO THE OLD SYMBOL TABLE,
AND THE CRITERION APPLIED TO X IS TRUE.

> describe preconditions of symbol-table-insert.

THE INPUT DOES NOT BELONG TO THE OLD SYMBOL TABLE.
```

Thus analysis by inspection has revealed some important additional restrictions which the user either was not clearly aware of, or in any case, did not explicitly state. The propagation of restrictions from the specifications of BUCKET-DELETE to SYMBOL-TABLE-DELETE and SYMBOL-TABLE-ADD could be achieved by the use of general reasoning mechanisms. However, these are such common specializations of the most general addition and deletion specifications that they are appropriately pre-compiled in the library.

CHAPTER THREE

OVERVIEW OF THE PLAN LIBRARY

3.1 Introduction

This chapter gives an overview of the plan library with an emphasis on taxonomy: English descriptions and example programs are used to give a feeling for the extent and overall organization of the knowledge in the library. Formal definitions for all library entries can be found in the appendix (see index for page numbers) written in a notation which is explained in Chapter Eight. Chapters Five, Six and Seven describe the use of the library in specific scenarios of analysis, synthesis and verification by inspection.

Methodology

My basic approach in developing a taxonomy of standard programming forms has been to start with the technical vocabulary commonly used and understood by experienced programmers, and then to apply my own intuitions to make appropriate generalizations and distinctions. I thus take the position that if programmers have evolved a name for something, it is probably an important concept. This means, for example, that there are plans in the library which capture the meaning of terms like "trailing pointer", "search loop" and "splice out".

Another method I have used to discover important programming concepts is to look for abstractions which unify the explanations of how many different programs work. For example, the concept of a directed graph makes it possible to express a number of standard algorithms independent of how the nodes and edges are represented in a particular program. This line of argument has also lead to including in the library a number of other familiar mathematical objects, such as functions, relations, sequences and sets.

Let me emphasize that the taxonomy represented in the current library is only intended to be a beginning. The exact contents of the current library has been determined primarily by the requirements of giving a complete account of one medium-sized example program, capturing all the important generalizations. The example program that was chosen for this is the symbol table program introduced in the scenario of Chapter Two. This particular program was chosen because it contains many different forms which are representative of common manipulations on symbolic data. I felt that a library which was adequate for this example would be a good start towards exploring the extent of this domain. I also felt that concentrating on one example in depth would lead to a better understanding of the relationship between different levels of abstraction, rather than touching on only the major points of many different programs.

Both of these intuitions have turned out to be good. Capturing all the important generalizations in this one program has touched upon a wide range of basic programming techniques. A complete account of the symbol table program has required filling the library with plans starting at a very abstract level, such as the idea of implementing a set as a discrimination function, down to the level of minor programming techniques, such as the use of flags to encode control information in binary valued data.

The small fraction of plans in the current library which are not directly motivated by the symbol table example fall into two categories. Some of these are obviously important basic plans which don't happen to be used in the example, such as counting and accumulation loops. Other plans are included to fill obvious gaps in the taxonomic structure of the library, such as the plan for splicing into a list (whereas only splicing out appears in the symbol table program).

Finally, while I do argue for the major outlines and organization of the current library, I do not expect that any reader will agree on every last detail. Many common manipulations on symbolic data are missing at present. The current library also needs to be expanded in many different directions, such as to include more general graph algorithms, matrix manipulations, and so on. However, it will hopefully be clear after reading this chapter where many of these extensions fit into the existing structure.

Implementation Relationships

A vocabulary of standard forms is not the only kind of knowledge involved in programming. A programmer also knows many ways of implementing one form in terms of others. The idea of implementing a set as a hash table, or of removing an entry from a list by splicing it out, are examples of implementation relationships (represented in the library by overlays). In building the library, the choice of programming vocabulary was often influenced by the implementation relationships. The motivation for making a vocabulary distinction was often to separate two cases which allow different implementations. For example, finite and infinite sets are distinguished in the library because membership tests in finite sets may be implemented by a loop which enumerates the elements, which is not a valid implementation for infinite sets. (The set of natural numbers is an example of an infinite set which is part of basic programming.)

An important kind of knowledge which is not yet explicitly represented in the library is the relative cost of various computations. However, I believe that in fact much of an expert programmer's knowledge about the relative cost of computations is embedded in his vocabulary. In other words, given that cost considerations are the primary motivation behind many standard programming ideas, the study of these ideas is a logical starting place for developing an understanding of computational cost. For example, the idea of a hash table is motivated by the desire to speed up various kinds of retrieval operations. This increase in speed is due to the fact that any single bucket in the table is smaller than the union of all the buckets. Future research will include studying the library further from this viewpoint in order to make this kind of knowledge more explicit.

Overall Organization

The current library contains approximately fifty input-output and test specifications, thirty data plans, and thirty temporal plans. These plans and specifications are organized in two ways: in a taxonomic hierarchy and by an interlocking network of approximately fifty overlays. There are two taxonomic relationships used in the library: specialization and extension. Note that a plan may be a specialization or extension of more than one other plan, so that the taxonomic hierarchy may be tangled.

A plan or specification is a *specialization* of another plan or specification if it has the same roles, but additional constraints. This means that the computations or data structures specified by the specialized plan are a subset of those specified by the more general plan.

A common motivation for introducing a specialization of a plan is because the properties of the specialization are exploited in some particular implementation. For example, consider the data plan, **Segment**, introduced in Chapter One. This data plan has three roles: a base sequence, an upper index, and a lower index. One way of implementing a mutable stack is to use an instance of Segment in which only the lower index is varied — the upper index is always equal to the length of the base sequence. This data plan is called Upper-segment; it is a specialization of Segment. Upper-segment has the same role names as Segment. Its constraints are the three constraints of Segment, i.e.

(i) The upper number is less than or equal to the length of the base sequence.
(ii) The lower number is less than or equal to the length of the base sequence.
(iii) The lower number is less than or equal to the upper number.

plus the following specializing constraint.

(iv) The upper number is equal to the length of the base sequence.

The basic idea of *extension* is to add an additional role to a plan or specification. The extended plan inherits all the constraints of the old plan.

A common kind of extension is to add an additional output to an input-output specification. For example, **Thread-find** is the standard input-output specification for finding a node satisfying a given criterion in a linear directed graph (thread). It has two input roles, named Input and Criterion, and one output role, named Output. The Output is a node of the Input thread which satisfies the Criterion predicate. When Thread-find operations are used in conjunction with other plans, such as splicing, it is convenient to have as output not only the node found, but also the previous node in the thread. This extension to Thread-find is called **Internal-thread-find**. Internal-thread-find has the same input roles as Thread-find, but *two* output roles, Output and Previous, with the additional constraint that Previous is the predecessor node of Output in the Input thread.

Object Types

Part of the hierarchy of object types is shown in Fig. 3-1. All the names in this figure are the names either of primitive object types or data plans. Similar figures later in this chapter will also include the names of input-output and test specifications, and temporal plans. Solid vertical lines between names in these figures denote specialization or extension relationships, with the specialized or extended plan always

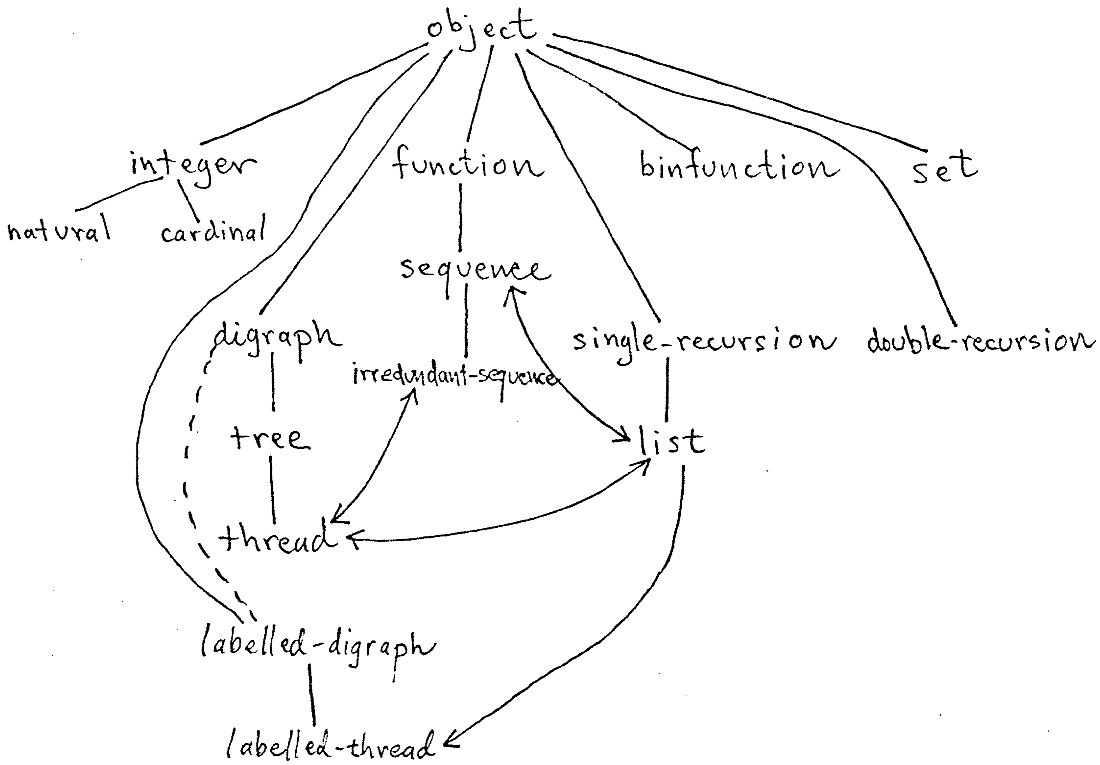

Figure 3-1. Hierarchy of Object Types.

below. Arrows in these figures represent overlays between plans. Most overlays are many-to-one mappings from instances of one plan to another. The arrow for such overlays points from the domain to the range. Overlays that are one-to-one are indicated by double-headed arrows. Dotted lines indicate "use" relations. For example, Labelled-digraph is defined using the definition of Digraph.

Referring to Fig. 3-1, note that the root node in the data object hierarchy is called **Object.** Below Object are the primitive types in the current library: **Integer**, **Function**, **Binfunction** (functions of two arguments), and **Set.** By "primitive" I mean here that systems which use the plan library are expected to have specific procedures for reasoning about these objects, and that this knowledge is not explicitly represented in the library itself.

The notion of Integer used here is a standard extension of the finite integers with a maximum element, infinity, and a minimum element, minus-infinity. Integer has specializations **Natural** and **Cardinal.** Instances of Natural are all the integers greater than or equal to one, not including infinity. Instances of Cardinal are all the integers greater than or equal to zero, including infinity.

Subsequent main sections of this chapter give overviews of parts of the library under the other main nodes in this hierarchy. There is a section about plans involving functions, one about plans involving sets, one about directed graphs, and one about recursive structures. However, these sections will not be able to discuss every plan in the library, since that would make the figures an unreadable clutter. For example, some plans involving minor programming techniques, such as the use of flags and various ways of implementing predicate tests are discussed as they arise in the later chapters (and their definitions can be found in the appendix.)

Notice the overlays in the middle of Fig. 3-1 between Sequence, List, Thread, and Labelled-thread. These overlays will be explained in more detail in subsequent sections. For now it is important just to point out this example of how multiple of points of view are catalogued in the library. Each of these data plans (Sequence is a specialization of the primitive object type Function) captures an alternative point of view on what could be called *linear structures.*

3.2 Functions

Fig. 3-2 shows the part of the plan library which involves functions. At the top left are three basic input-output specifications which have functions as inputs or outputs. **@Function** is the specification for applying a function to an argument to get a value.[1]

Another common operation performed on functions is to change the value associated with a given argument. The input-output specification for this operation is called **Newarg.** Newarg has three inputs: the old function, an argument, and the new value. The output is a new function such that the given argument maps to the new value and the values of all other arguments remain unchanged.

A less commonly used specification is **Newvalue.** Newvalue also has three inputs: the old function, an old value, and a new value. The output is a new function such that all the arguments that used to map

1. The character "@" is intended to be read as "apply".

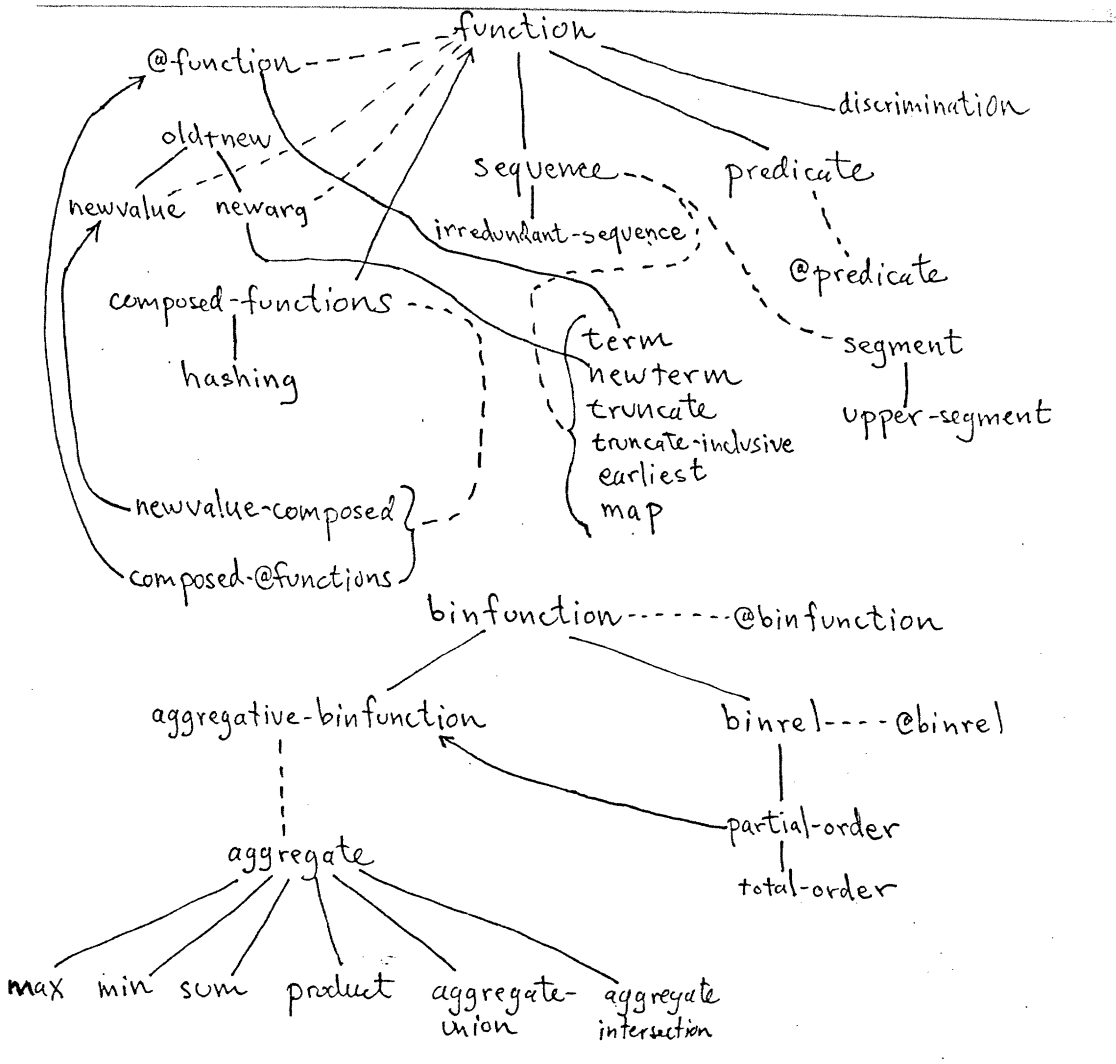

Figure 3-2. Plans Involving Functions.

to the old value now map to the new value and the values of other arguments remain unchanged. Newvalue will be used as part of the analysis of operations on hash tables.

Notice that these specifications make no commitment as to whether the old function is copied or modified to get the new function. The copying and side effect versions will be treated as specializations. The input-output specification, **Old+new**, of which Newarg and Newvalue are extensions, is a very general form which makes it possible to state this idea in general. It is advantageous to work with these more abstract specifications as much as possible, since they unify the logical structure of a larger number of programs. These same remarks apply to all other input-output specifications in this chapter which are shown as extensions of Old+new. Plans involving side effects are discussed further in Chapter Eight.

At the middle left of Fig. 3-2 are some plans having to do with implementing a function as the composition of two functions, i.e. by the data plan **Composed-functions.**

Composed-@functions is a temporal plan for implementing @Function for a function implemented as Composed-functions, i.e. apply the second function of the composition to the output of applying the first function to the given argument.

The plan **Newvalue-composed** and the overlay between it and Newvalue express the fact that a Newvalue operation on a function implemented as Composed-functions can be implemented by a Newvalue operation on the second function of the composition alone. This plan arises in the analysis of the symbol table example, where the hash table is viewed as the composition of two functions: a numerical hash function which doesn't change, and an array that is modified to insert new entries.

Notice that the data plan **Hashing** is a specialization of Composed-Functions. As we have seen in the scenario, the first function in this case is referred to as the hash function, and the second (a sequence) is referred to as the table. A discrimination function can be implemented as a hash table, in which case the table is a sequence of sets, called the buckets. The utility of this implementation is that changes (e.g. Newvalue operations) to a discrimination implemented this way may be achieved by changing only the table, as specified by the Newvalue-composed plan discussed above. Discrimination functions will be discussed further in the next section on sets.

Sequences

Sequences are viewed formally as functions on the natural numbers which are defined on some initial interval (up to the length of the sequence) and undefined elsewhere. A common specialization is **Irredundant-sequence**, i.e. sequences in which no two terms are equal.

A number of common operations on linear structures are most naturally specified in terms of sequences. Fig. 3-2 shows several such input-output specifications. The first two specifications, **Term** and **Newterm**, are simply specializations of @Function and Newarg to the case when the functions involved are sequences.

The next two specifications have to do with truncating sequences according to some criterion (a predicate). In both cases a precondition is that there exist some term of the input sequence which satisfies the criterion. The output sequence in both cases is a finite initial subsequence of the input sequence. In the case of **Truncate-inclusive**, all but the last term of the output sequence fail the criterion; the last term passes. In the case of **Truncate**, all terms of the output sequence fail the criterion and the length of the output sequence is one less than the index of the first term in the input sequence that passes the criterion.

A closely related input-output specification is **Earliest**. Again the inputs are a sequence and a criterion, and a precondition is that there exist some term of the input sequence which satisfies the criterion. The output is the earliest term of the sequence which passes the criterion, i.e. all terms with indices lower than the index of the output fail the criterion.

The last input-output specification on sequences in Fig. 3-2 is **Map**. Its input and output are sequences of the same length. An additional input (Op) is a function such that each term of the output is the result of applying that function to the corresponding term of the input.

Aggregations

This section introduces some simple algebraic structure which captures the similarity between programs which compute sums, products, set unions and intersections, maximums and minimums. The input-output specification which is the generalization of all these operations is called **Aggregate.** Aggregate takes as input a (non-empty, finite) set of objects and a function of two arguments which is commutative, associative and has identity elements. Such a function is called an **Aggregative-binfunction.** (If the function also has an inverse, then it is an Abelian group.) The output of Aggregate is the result of composing the application of the aggregative function to the members of the input set. The algebraic properties of aggregative functions guarantee that the order of this composition doesn't matter.[1]

Fig. 3-2 also names six common specializations of Aggregate for particular aggregative functions: **Sum** (Plus), **Product** (Times), **Aggregate-union** (Union), **Aggregate-intersection** (Intersection), **Max** (Greater), and **Min** (Lesser).

Relations

Relations are treated formally as boolean valued functions. A **Predicate** is a boolean valued function of one argument; a **Binrel** is a boolean valued function of two arguments. Correspondingly, **@Predicate** is the specialization of @Function to predicates, and **@Binrel** is the specialization of @Binfunction to binary relations.

Note in Fig. 3-2 the overlay between **Partial-order** and Aggregative-binfunction. This overlay allows the following code

```
(COND ((> N MAX)(SETQ MAX N)))
```

to be analyzed as an application of the Lesser function , which then allows a loop with this code in the

1. Which is why the input is a set rather than a list or sequence. Also there is some subtlety being suppressed here concerning whether the input should be a set or a multiset. In the case of union, intersection, maximum and minimum, the occurrence of duplicates doesn't matter, and therefore the set abstraction is definitely appropriate. Sum and product, however, do not have this property. Nevertheless, I argue that, conceptually, the input to a summation operation is a _set_ of objects in the sense that even though viewed as integers they may have the same behavior, they represent conceptually distinct quantities and are therefore not identical. See Chapter Eight for more on the notion of behavior versus identity.

body to be analyzed as the implementation of the Min operation (and similarly, when the test is "<", the implementation of Max).

3.3 Sets

Fig. 3-3 shows part of the plan library which involves sets. At the left of the figure we have first some common input-output and test specifications with sets. **Member?** tests whether a given object is a member of a given set. **Any** is a more complicated test: given a set and a predicate as inputs, it succeeds if there exists a member of the set which satisfies the predicate, and returns such a member as its output; otherwise it fails. **Set-find** is a related input-output specification: it has the precondition that there that there exists a member of the input set which satisfies the input predicate, and simply returns such a member as its output.

The next two input-output specifications each have a set as input and a set as output. **Each** is a specification used to analyze programs like `(MAPCAR 'SQRT L)`, where the input list, `L`, is viewed as a set and `SQRT` is a function applied to each element of the set to get an output set. **Restrict** takes as input a set and a predicate and returns the subset which satisfies the predicate. As in the case of functions, no commitment is made in these specifications to whether the old set is copied or modified to get the new set.

Finally, **Set-add** and **Set-remove** specify addition of a given object to a set and removal of a given object from a set. The very abstract specification **Old+input+new-set**, of which both Set-add and Set-remove are specializations, captures what the implementations of these specifications have in common.

The implementation of sets is a very rich area of programming technique [62]. It is not the goal here to be exhaustive of all of the possibilities, but rather to show by example how to go about formalizing such implementations using the plan calculus. In addition to the standard simple implementations of sets as sequences and lists, this section presents two examples of non-trivial set implementations which are involved in understanding the the symbol table program.

The overlay for viewing a list as the implementation of a set is recursively defined: an object is a member of the implemented set iff it is the head of the list or it is a member of the set implemented by the tail of the list. The empty set is usually implemented by Nil. There are also overlays in the library for viewing Push and Pop operations as Set-add and Set-remove operations. The implementation of other set operations is more naturally expressed taking the point of view of the list as a directed graph, which will be discussed in the next section.

Discrimination

One basic idea underlying many set implementations is the use of a function (called a **Discrimination**), whose range is a set of sets (called buckets). Such a function can be viewed as implementing a set wherein a given object is a member iff it is a member of the bucket obtained by applying the discrimination function to that object. This is the basic "divide and conquer" strategy underlying both hash tables and discrimination nets.

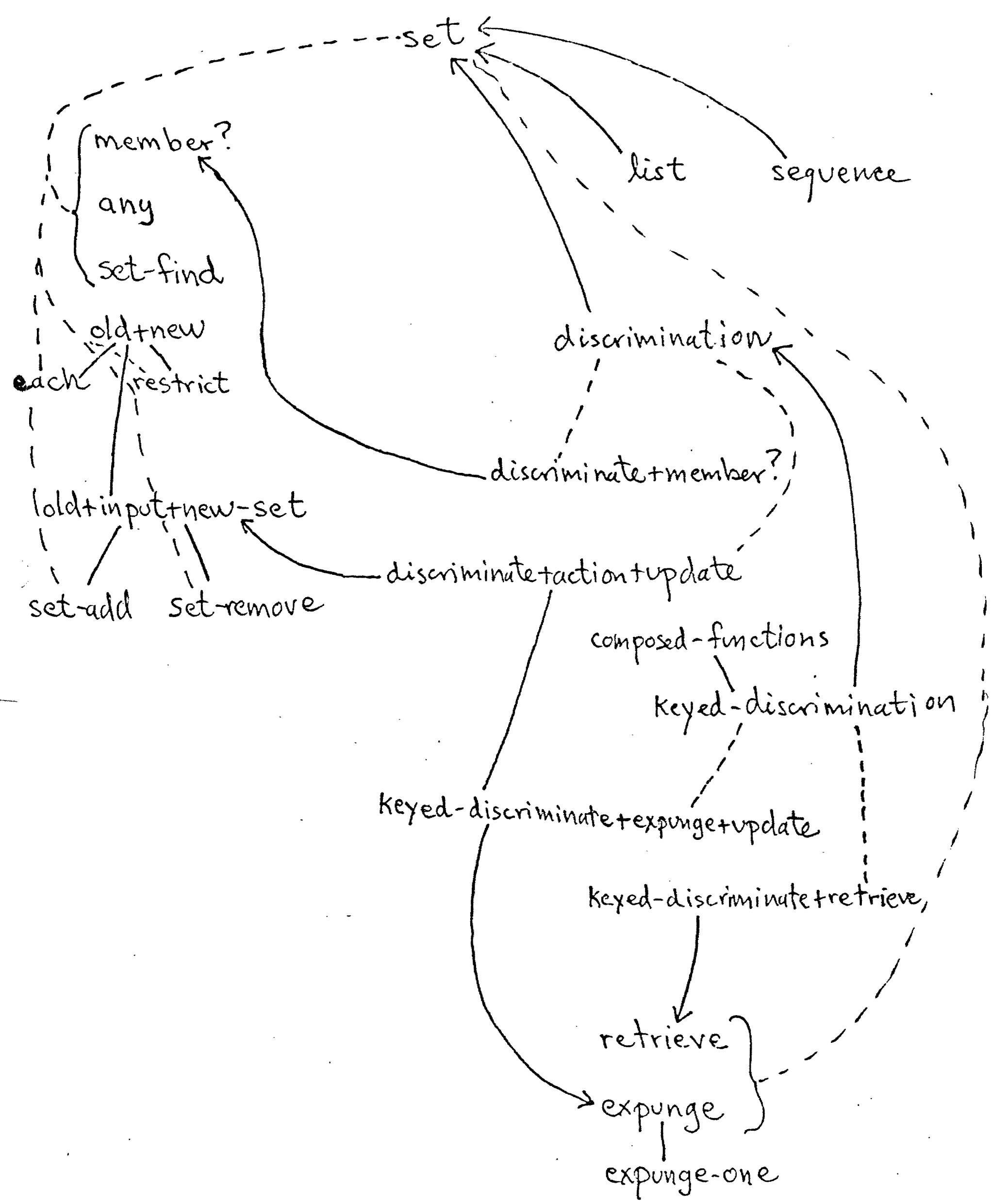

Figure 3-3. Plans Involving Sets.

Testing for membership in a set implemented as a discrimination is implemented by the two step plan **Discriminate+member?**. The first step is to apply the discrimination function to the given object to determine which bucket to look in. The second step is an instance of Member?, with the input set being the bucket fetched by the first step. Since any single bucket in a discrimination is smaller than the overall implemented set, (except in the case of a degenerate discrimination function which maps all objects to a single bucket), this implementation leads to a increase in speed at the cost in space for encoding the discrimination function.

Both Set-add and Set-remove for input and output sets implemented as discriminations are implemented by specializations of the same three step plan: first, apply the discrimination function to the input object to obtain a bucket; second, perform the appropriate operation on that bucket to get a new bucket; and third, update the discrimination function so that all domain objects which used to map to the old bucket now map to the new bucket (i.e. a Newvalue operation). These three steps are expressed by the **Discriminate+action+update** plan.

Associative Retrieval

Associative retrieval adds to basic set operations the concept of a key. The function which associates members of a set with keys is called the key function. Given a set, such as the entries in a symbol table, we are often more interested in finding a member with a given key, than in just testing for membership. The most basic specification for associative retrieval is called **Retrieve** (see bottom of Fig. 3-3). Given a set, a key function and an input key, Retrieve has two cases: if there exists a member of the set with the given key, then it succeeds, and its output is such a member; otherwise it fails. The other common associative retrieval specification, **Expunge**, removes all members of an input set which have a given key. **Expunge-one** is a common specialization of Expunge which often allows a simpler implementation. Expunge-one has the additional precondition that there exists exactly one member of the input set with the given key.

Keyed Discrimination

To speed up associative retrieval for a given key function, a discrimination function can be used which is itself the composition of two functions. This is the data plan **Keyed-discrimination** (see middle of figure). The first function is the key function. The second function, called the bucket function, maps from the set of keys to the buckets. In typical usage, the bucket function may itself be decomposed further into a Hashing (or another keyed discrimination, as will be discussed shortly).

The implementation of Retrieve from a keyed discrimination has the same two step structure as the implementation of Member? for a discrimination: first, apply a function to obtain a bucket; second, perform the appropriate operation on the bucket. In the case of a keyed discrimination, however, the appropriate bucket is obtained by applying the bucket function (which is the second half of the composed functions which implement the discrimination) to a given key, instead of applying the full discrimination function to an object which might be a member of the set. This plan is called **Keyed-discriminate+retrieve.**

For Set-add and Set-remove, the fact that a discrimination is further implemented as a keyed discrimination makes no difference.

Associative deletion (Expunge) from a keyed discrimination is implemented by a three step temporal plan, **Keyed-discriminate+expunge+update**, which is an extension of the Discriminate+action+update plan described earlier (see figure). Keyed-discriminate+expunge+update has the following three steps. (This plan is used in the analysis of the symbol table deletion example.)

(i) First, the appropriate bucket is obtained by applying the bucket function of the keyed discrimination to the given key.

(ii) Then, just as in Discriminate+action+update, the action on the whole set reduces to a corresponding action on the bucket. The Action step here is an instance of Expunge.

(iii) The final Update step is similarly a Newvalue operation on the discrimination function so that all domain objects which used to map to the old bucket, map to the new bucket. Furthermore, in the case of a keyed discrimination, only the bucket function needs to be updated; the key function stays unchanged.

The idea of keyed discrimination can be generalized to multiple key data bases in two ways. One approach is to have separate discrimination functions for each key function which map into a shared set of buckets. Associative retrieval on a pattern of keys is then implemented by intersecting the appropriate buckets. (This is the idea underlying the implementation of the Conniver data base [45].) Alternatively, the discrimination functions for different keys can be composed, so that each function maps to a bucket which is itself a set implemented as a discrimination on the next key. This is the basic idea underlying discrimination nets.

3.4 Directed Graphs

Directed graphs are one of the most common programming data structures. A **Digraph** is defined formally in the library as a set of nodes and an edge relation. For example, a Lisp list may be viewed as a directed graph wherein the nodes are Lisp cells, the edge relation is Cdr, and Car is a function which attaches a label to each node. The nodes of a standard Lisp binary tree structure may also be viewed as a directed graph in which the edge relation is the union of the Car and Cdr relations between the nodes. This view is particularly appropriate for programs which splice objects in and out of lists or trees.

Barstow [6] has recently developed a set of rules for generating many standard programming algorithms for operating on directed graphs in the general case. Some time in the future his rules should be incorporated into the present library. This section concentrates on the special case of acyclic graphs with a single root, i.e. trees, and furthermore on the linear case of trees, which are here called *threads*.

Fig. 3-4 shows some standard specializations of Digraph. **Tree** is a directed graph in which there is

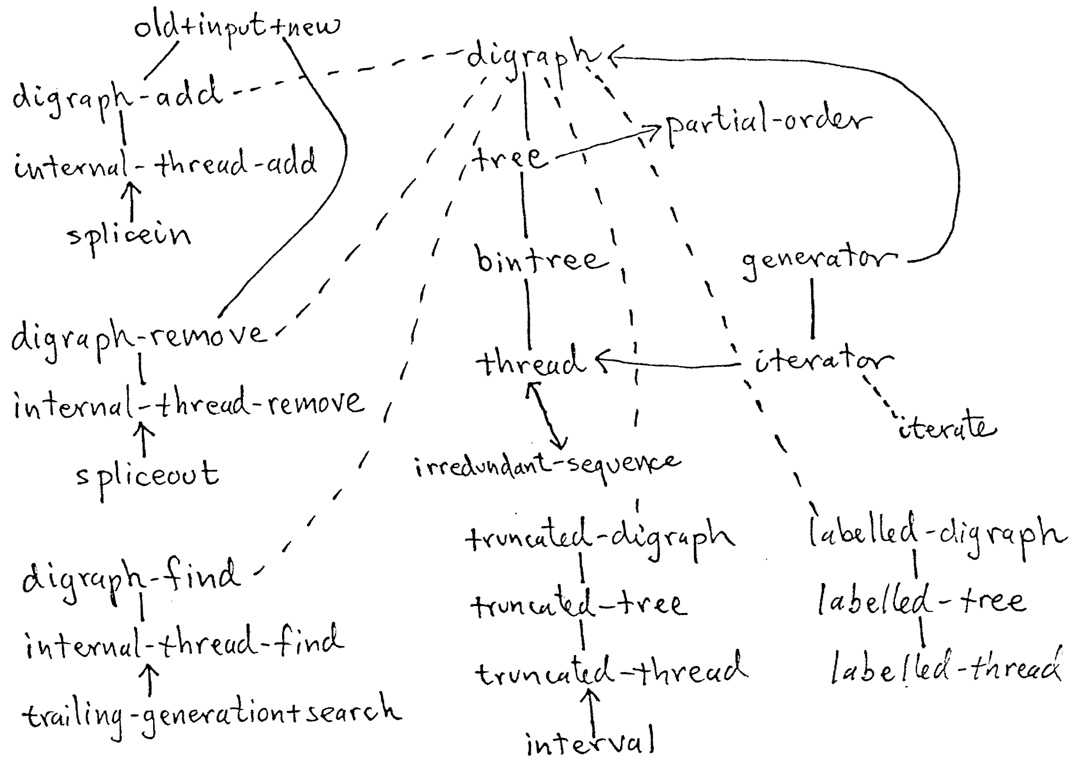

Figure 3-4. Plans Involving Directed Graphs.

a root and no cycles.[1] A **Bintree** is a a tree in which each node is either a terminal or it has exactly two successors. A **Thread** is a specialization of Tree in which the successor of each node is unique. This also means that the predecessor of each node in a thread (if it exists) and the terminal node are unique.

The vocabulary of partial orders is often applied to trees and threads. For example, it is common to think of a nodes in a tree or thread being "before" other nodes. This viewpoint is formalized by an overlay from Tree to Partial-order indicated in Fig. 3-4. A tree is viewed as a partial order in which two nodes are less than or equal iff they are successor* (the transitive closure of the successor relation) in the tree or are the same node. The root of the tree in this view becomes the minimum element of the partial order. Furthermore, if the tree is a thread, then the partial order is total.

Fig. 3-4 also shows an overlay between Irredundant-Sequence and Thread. An irredundant sequence can be viewed as a thread in which the first term of the sequence corresponds to the root of the thread and any two consecutively numbered terms in the sequence are successors in the thread. Notice also that this overlay is one-to-one, which means that for each instance of Thread there is a unique corresponding instance of Irredundant-Sequence, and vice versa. This allows us to use both the standard vocabulary of sequences (such as length and the idea of the n-th element) and of directed graphs (such as the idea of successors) as appropriate to specify properties of linear structures.

Generators

One of the most common ways of implementing directed graphs in programming is to specify a single node (called the "seed") and a binary relation such that the nodes of the desired graph are the transitive closure of the given node under the given relation. This implementation is captured by the data plan **Generator.**

Iterator is the specialization of Generator which generates threads. This constrains the binary relation of an iterator to be many-to-one (i.e. a function) and to have no cycles within the transitive closure of the seed. This data plan is used in the analysis of counting loops and loops which CDR down a list. The effect of the generating part of such loops is abstracted further in terms of the input-output specification **Iterate**, which takes an iterator as input and outputs the sequence of generated nodes. Loop plans and temporal abstraction will be discussed further in the next section.

Truncated Directed Graphs

Another common way of specifying a directed graph is as part of another directed graph. This is particularly used for specifying finite parts of infinite graphs such as intervals of the natural numbers.

The most general data plan describing this technique is **Truncated-digraph**. This data plan has two roles: the Base graph and a Criterion predicate. The criterion must divide the nodes of the base graph into three sets: a set of boundary nodes which satisfy the criterion; interior nodes, from which boundary

---

1. Notice that this definition of tree does not constrain a node to have a unique predecessor, i.e. there can be sharing of substructure in the tree. In later versions of the library it will be necessary to distinguish between acyclic rooted directed graphs in which nodes do and do not have unique predecessors.

nodes can be reached (in a finite number of successor steps); and exterior nodes, which can be reached from boundary nodes. When the base graph is a thread (**Truncated-thread**), this means more simply that some node of the thread (either the root or a finite successor of the root) satisfies the criterion. Each such criterion thus determines a finite subgraph of interior nodes, either including or not including the boundary nodes.

Examples of truncated directed graphs in Lisp programming are Cdr threads truncated by the Null predicate and Car-Cdr binary trees truncated by the Atom predicate.

A closely related way of specifying truncated threads is in terms of upper and lower bounds on some total order. This is called an **Interval.** For example, the integers from 10 to 100 are specified as an instance of Interval in which the total order is Le, the lower bound is 10, and the upper bound is 100.

Splicing Plans

Thinking in terms of directed graphs is particularly appropriate for understanding programs which add or remove nodes in the middle of lists or trees. This section introduces a number of plans related to adding or removing internal nodes of threads in particular. These plans are used for example in analyzing the symbol table deletion program.

At the left of Fig. 3-4 are some basic input-output specifications on directed graphs which are involved in understanding splicing plans. **Digraph-add** is the basic specification for adding a node to a directed graph. It takes an old graph and a node as inputs and gives a new graph as output. All that can be said at this level of abstraction is that the input is a node of the new graph, and that all the successor relationships in the graph not involving either the added node, its predecessors or successors remain unchanged. Digraph-add does not specify where in the directed graph the node is to be added. **Internal-thread-add** is a specialization of Digraph-add in which the old and new graphs are threads and the new node is added anywhere but at the root.

The basic input-output specification for removing a node from a directed graph is **Digraph-remove.** Like Digraph-add, it takes an old graph and a node as input, and returns a new graph. All the successor relationships in the directed graph not involving the removed node remain unchanged. The successors of the removed node in the old graph become the successors of the predecessor of the removed node in the new graph. **Internal-thread-remove** is the specialization of Digraph-remove in which the old and new graphs are threads and the node to be removed is not the root.

Programs which splice nodes in or out of a thread typically have two steps. The first step is to find the place in the thread where the addition or removal is to occur. The output of this step is usually a pair of successor nodes, such that either the new node is to be added between them or the second node is the one to be removed. If the thread is implemented as an iterator, the second step is then to modify the generating function so as to either splice in or splice out a node, as the case may be.

The input-output specification of the first step (finding internal nodes), which is shared between add and remove programs, is called **Internal-thread-find.** Given a thread and a criterion, Internal-thread-find returns a node of the thread (other than the root) which satisfies the criterion, and its predecessor. The typical implementation of this specification is to use a search loop which keeps track of both the current and the immediately preceding node. This loop pattern is captured by the recursive temporal plan Trailing-generation+search, which will be discussed further in the next section.

The second step implementing removal of a node is a Newarg operation in which the association between the node to be removed and its predecessor is modified to be an association between the predecessor and the successor of the node to be removed. For example, in the BUCKET-DELETE program of the scenario in Chapter Two, the node to be removed is in P and its predecessor is in Q; the generating function is CDR. The code for splicing out in BUCKET-DELETE is as follows.[1]

```
(RPLACD Q (CDR P))
```

The plan for this form of code in general is called **Spliceout.**

The second step implementing addition of a node requires two Newarg operations: one to make the new node point to its successor, and one to make the predecessor of the new node point to it. For example, addition of a node to a Lisp list iterator might be coded as follows.

```
(RPLACD NEW CURRENT)
(RPLACD PREVIOUS NEW)
```

The plan for this form of code in general is called **Splicein.**

The last data plan in Fig. 3-4 to be discussed is **Labelled-digraph.** This data plan has two roles: Spine (a digraph) and Label (a function on the nodes of that graph). An important specialization is **Labelled-thread,** in which the spine is further constrained to be a thread. This plan is used to view a Lisp list as a Cdr thread with objects attached at each node by Car. As discussed above, this view is particularly natural for understanding programs which modify lists by splicing.

3.5 Recursive Plans

Recursively defined plans are used in the plan calculus to represent unbounded structures. A recursive plan is one in which one or more roles are constrained to be instances of the plan itself. This section will discuss only the special case of singly recursive plans, since the plans and overlays for doubly and multiply recursive structures tend to be long and more detailed than those for singly recursive structures, without introducing any fundamentally new ideas.

At the top of the hierarchy of recursive plans in Fig. 3-5 is a minimal plan, **Single-recursion,** which says nothing more than that there is a role, Tail, constrained to be either an instance of Nil or itself a Single-recursion. **Nil** is a distinguished object used to terminate singly recursive structures.

The most important singly recursive data plan, List, will be discussed first in the following section. Singly recursive temporal plans, e.g. loops, will be discussed in the section following that. Finally, *temporal abstraction* will be introduced as a point of view which links singly recursive temporal plans with singly recursive data plans. Chapter Nine treats loops and temporal abstraction in much more detail.

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1. RPLACD is modelled as Newarg, where the first argument to RPLACD is the domain element and the second argument is the new range element.

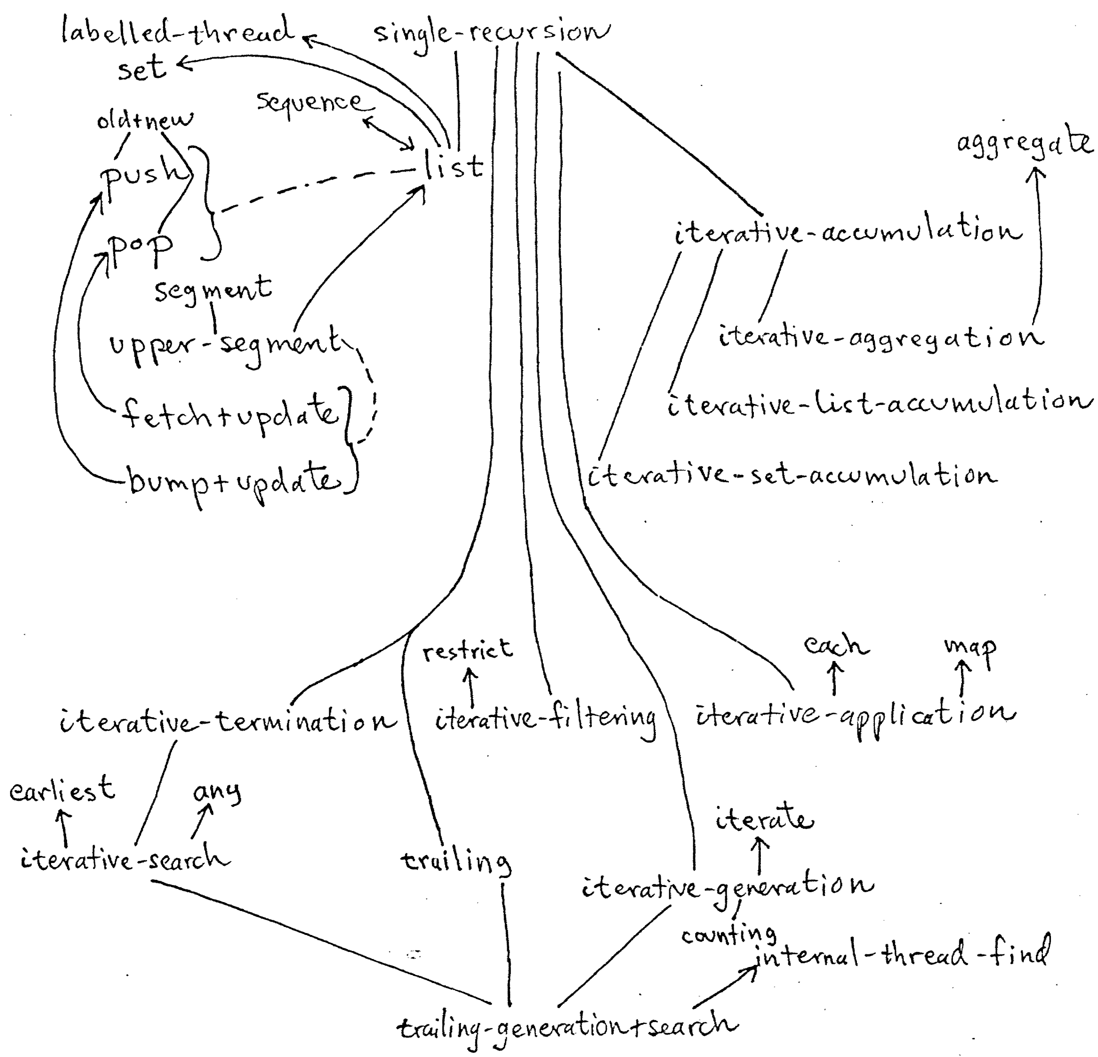


Figure 3-5. Recursive Plans.

Lists

List is a singly recursive data plan with two roles, Head and Tail. The head may be any object, but the tail must be an instance of List or Nil. It is important not to think of this data plan too concretely. The List plan is trying to capture what all recursive views of data structures have in common. List is the point of view which is used for making (linear) inductive arguments about data structures. Thus the reader should not identify the data plan List too closely with, for example, the Lisp list. Think of the data plan List as if it were called "singly recursive data structure".

Two basic input-output specifications on lists are shown at the top left of Fig. 3-5. **Push** takes as input a list (or Nil) and an object, and returns a new list, whose tail is the input list and whose head is the input object. **Pop** takes a list and returns its head and tail as its two outputs.

A common implementation of lists is to use a sequence (e.g. an array) with an index to where the current head is stored. The data plan which captures this implementation is called **Upper-segment.** This plan is a specialization of **Segment**, which has three roles: the Base, which is a sequence, and the Upper and Lower bounds, which must be valid indices for the base. Upper-segment is a specialization of Segment in which the upper bound is equal to the length of the base sequence. Push and Pop operations on this implementation are implemented by the two-step temporal plans, **Bump+update** and **Fetch+update**, respectively. The second step in each of these plans is either to add or subtract one from the old lower bound to get a new lower bound.[1] The first step in implementation of Push is a Newterm operation, which makes the given object the head of the new list. The first step in the implementation of Pop is a Term operation, which fetches the current head of the list.

Multiple Views of Linear Structures

Fig. 3-5 also indicates overlays between lists and other linear structures, such as sequences and threads. For example, whether a given data structure is viewed as a list or as a sequence depends on what we want to say about it. Certain properties are easier to specify inductively, in which case the list view is appropriate. In other cases, explicit quantification over the indices of a sequence is more convenient. In the overlay between List and Sequence, the head of the list corresponds to the first term of the sequence, and the head of the n-th tail of the list corresponds to the $(n+1)$th term of the sequence.

In the overlay between List and Labelled-thread, the nodes of the spine of the thread are the list and all of its tails. The edge function on the nodes of the spine is the Tail function, and the label function is Head. Thus we now have two ways of viewing Lisp cells which have Lisp cells or Nil as their Cdr. We can view such a Lisp cell as implementing a list in which the Car of the cell is its head and the Cdr is its tail; or we can view the same Lisp cell as the seed for generating a Cdr thread which is labelled by Car.

1. Again, at this level of abstraction no commitment is made in these plans as to whether the instance of Upper-segment is modified by side effect or copied. These are treated as specializations, just as the "pure" and "impure" versions of Push and Pop are treated as specializations.

Linear structures may also be viewed as (i.e. implement) sets. In particular, a list may be viewed as the set whose members are the head of the list unioned with the tail of the list viewed as a set. Nil is usually viewed as the empty set. In this view, neither the order of occurrence of elements in the list nor the occurrence of duplicates matters. In this view, Push and Pop operations on a list are implementations of Set-add and Set-remove operations. Alternatively, viewing lists as labelled threads, Set-add and Set-remove may be implemented by Splicein and Spliceout plans. Both of these points of view are needed to understand how entries are added and removed in the symbol table example: in SYMBOL-TABLE-ADD entries are added to the bucket by a Push operation (implemented in Lisp by CONS); in BUCKET-DELETE entries are removed by a Spliceout plan (implemented in Lisp using RPLACD).

Loops

The taxonomy of loop structures used in the library is based on Waters' [73] method for analyzing loop programs. Waters' method decomposes loops into fragments which correspond to "easily understood stereotyped fragments of looping behavior." The next section describes overlays which allow these fragments to be logically composed, rather than interleaved (as they are in an unanalyzed loop), which makes their net effect easier to understand. For example, consider the following program, which sums up the non-nil elements of a list.

```
(DEFINE SIGMA
   (LAMBDA (L)
      (PROG (S N)
            (SETQ S 0)
         LP (COND ((NULL L)(RETURN S)))
            (SETQ N (CAR L))
            (COND (N (SETQ S (PLUS S N))))
            (SETQ L (CDR L))
            (GO LP))))
```

Waters distinguishes three types of fragments (he calls them *plan building methods*) in loops with one exit test. The first type he calls "basic loops". A basic loop is characterized by the fact that all of the computation in the body of the loop can potentially affect the termination of the loop. For example, the basic loop part of SIGMA is the following.

```
(LAMBDA (L)
   (PROG (...)
         ...
      LP (COND ((NULL L)...))
         ...
         (SETQ L (CDR L))
         (GO LP)))
```

Basic loops are further decomposed into a generation part (e.g. the part involving CDR above) and a termination part (e.g. the NULL test above). The temporal plan which captures the form of the generating part of loops in general is called **Iterative-generation**. The plan which captures the form of single exit tests is called **Iterative-termination**. Both of these are extensions of Single-recursion (see Fig. 3-5). The advantage of this further decomposition is it allows us to capture the similarity between loops which have the same generation part but different terminations. For example, one can form many different loops with **Counting** as the generation part, but with different terminations. (Counting is a a specialization of Iterative-generation in which the generating function is Oneplus)

Waters' second category of plan building method is called "augmentations". Augmentations are characterized by the fact that they consume values produced by other parts of the loop and produce values which may be used by other augmentations. In the library, augmentations are further divided into *application* and *accumulation*. The distinction between these two types of augmentations rests on whether there is any "feedback", i.e. whether the augmentation consumes its own values from previous iterations — accumulation does, application does not. For example, the following is the application part of SIGMA.

```
(PROG (...N)
      ...
   LP ...
      (SETQ N (CAR L))
      ...
      (GO LP))
```

The plan for this form of code in general is called **Iterative-application.** SIGMA also has an example of accumulation, as shown below.

```
(PROG (S...)
      (SETQ S 0)
   LP ...(RETURN S)...
      ...(SETQ S (PLUS S ...))...
      ...
      (GO LP))
```

The plan for this form of code in general is called **Iterative-accumulation.** Three common specializations of Iterative-accumulation are shown in Fig. 3-5. **Iterative-set-accumulation** is a specialization in which the accumulation operation (e.g. PLUS above) is Set-add and the initial accumulation is the empty set. **Iterative-list-accumulation** is a specialization in which the accumulation operation is Push and the initial accumulation is Nil. **Iterative-aggregation** is a specialization in which the accumulation operation is the application of an aggregative function (as discussed earlier in the section on functions) and the initial accumulation is the identity element for that function.

Waters' final type of plan building method is called "filtering". It is the special case of an augmentation whose body is a conditional. The purpose of filtering usually is to restrict the values that will be consumed by some other augmentation. For example, in SIGMA the following is the filtering part of the loop which restricts the accumulation part to consuming only the non-nil inputs.

```
(PROG (...N)
      ...
   LP ...
      (COND (N ...)
      ...
      (GO LP))
```

The plan for this form of code in general is called **Iterative-filtering.**

Finally, the **Trailing-generation+search** plan at the bottom of Fig. 3-5 illustrates an important feature of the taxonomy in the library, namely that it is a *tangled* hierarchy. Trailing-generation+search combines the features of three plans. One of these plans is Iterative-generation, an example of which is the following.

```
(PROG (P ...)
      ...
   LP (SETQ P (CDR P))
      ...
      (GO LP))
```

The second plan is **Iterative-search.** Iterative-search is a specialization of Iterative-termination wherein the exit test is the application of a predicate which doesn't change as the computation proceeds, and in which the final object which satisfied the exit test is available outside the loop. This plan is suggested by the following code.

```
(PROG (P ...)
      ...
   LP ...
      (COND (...P...
             ...P...
             (RETURN ...)))
      ...
      (GO LP))
```

The final plan is **Trailing,** which captures the idea of keeping track of the immediately previous value of some loop variable, as suggested by the following code.

```
(PROG (P Q)
      ...
   LP (SETQ P ...)
      ...
      (SETQ Q P)
      (GO LP))
```

Tailing-generation+search inherits the roles and constraints of all three of these plans. For example, the combination[1] of the three example fragments above gives the essential loop structure of BUCKET-DELETE, as shown below.

```
(PROG (P Q)
      (SETQ Q BUCKET)
   LP (SETQ P (CDR Q))
      (COND ((EQUAL (CAAR P)) INPUT)
             (RPLACD Q (CDR P))          ;SPLICE OUT.
             (RETURN BUCKET))
      (SETQ Q P)
      (GO LP))
```

1. The code fragments above cannot literally be combined to get the loop of BUCKET-DELETE. The appropriate domain for this combination is the plan calculus.

Temporal Abstraction

The basic idea of temporal abstraction is to view all the objects which fill a given role at each level in a recursive temporal plan as a single data structure. In programming language terms, this often corresponds to having an explicit representation for the sequence of values taken on by a particular variable at a particular point in a loop. This idea is also present in the work of both Waters [73] and Shrobe [64]. Using temporal abstraction, the recursively defined plan for a loop can be viewed much more simply as a simple composition of operations on sequences or sets. Chapter Nine explains how this analysis is formalized using overlays for the various loop plans described in the preceding section.

Fig. 3-5 shows some of these overlays. For example, Iterative-generation can be temporally abstracted as Iterate. The input to Iterate in this overlay is an iterator whose seed is the initial value of the relevant loop variable (e.g. P above) and whose generating function is the function applied each time around the loop (e.g. CDR above). The output of Iterate corresponds to the sequence of values taken on by the loop variable.

The relationship between the sequences of values consumed and produced in an instance of Iterative-application can similarly be viewed as a Map operation. In programs where order and occurrence of duplicates in the loop values doesn't matter, a further temporal abstraction can be made by viewing the values consumed and produced as sets. In this view, Iterative-application implements Each.

Similarly, Iterative-search can be viewed as implementing either Earliest or Any, depending on whether the inputs over time to the exit tests are viewed as a sequence or a set; and Iterative-filtering can be viewed as the implementation of Restrict.

CHAPTER FOUR

THE PLAN CALCULUS

4.1 Introduction

The purpose of this chapter is to give an intuitive definition of the plan calculus. (A formal definition is given in Chapter Eight.) Practically speaking, the plan calculus is a network-like formalism. This chapter introduces a diagram notation which will be used to define and describe the use of plans in succeeding chapters. There are many well-known ways of storing such networks in a computer to facilitate various kinds of updating and retrieval. Concrete storage representations of the plan calculus will therefore not be discussed here. Several different concrete storage representations have been implemented and used by the author, Shrobe [64] and Waters [72].

The plan calculus has two major components: plans and overlays. The first part of this chapter introduces plan diagrams, followed by a discussion of the relationship between such diagrams and the Lisp code for a program. The second part part of this chapter introduces overlay diagrams, followed by some general observations on the use of overlays as a preview of coming chapters.

Side effects and mutable objects will only be mentioned in passing in this chapter, since a proper discussion requires the formal foundations developed in Chapter Eight. Plans involving side effects are also discussed further in Chapter Eight.

4.2 Plans

The basic idea of a plan in the plan calculus comes from an analogy between programming and other engineering activities [54]. "Plans" of various kinds are used by many different kinds of engineers. For example, an electrical engineer uses circuit diagrams and block diagrams at various levels of abstraction; a structural engineer uses large-scale and detailed blue prints which show both the architectural framework of a building and also various subsystems such as heating, wiring and plumbing; a mechanical engineer uses overlapping hierarchical descriptions of the interconnections between mechanical parts and assemblies.

A fundamental characteristic shared by all these types of engineering plans is that at each level there is a set of *parts* with *constraints* between them. Sometimes these parts correspond to discrete physical components, such as transistors in a circuit diagram, but more often the decomposition is in terms of function. For example, a simple amplifier in an electrical block diagram has the functional description $V_2 = kV_1$, where V_1 and V_2 are the input and output signals, and k is the amplification factor. As far as this level of plan is concerned the amplification may be realized in any number of ways. A primitive component may be used or another plan may be provided which decomposes the amplifier further.

By analogy, plans in programming specify the parts of a computation and constraints between them. In the plan calculus, the names of the parts of a computation are called *roles*. It is natural to think of roles as selector functions. For example, consider the Segment plan discussed in Chapter One, which has three roles named Base, Upper and Lower. To refer to the Base sequence of this plan we write Segment.Base, to refer to the Upper index we write Segment.Upper, and so on. The point (".") in this notation has the same intuitive meaning as in its use for selecting fields of record structures in programming languages such as PL/1.

An expression with a point in it is called a *path* name. If a role is filled by an instance of another plan, the point notation can be used several times. For example, consider a plan named Bump+update which has a role named Old, constrained to be a Segment. The path name Bump+update.Old.Upper then refers to the upper index of the Old segment of the plan.

All composite plans are composed (using roles and constraints) out of three primitives types: *input-output* specifications, *test* specifications and primitive *object* types (integers, sets and functions). Plans composed up exclusively out of objects are called *data plans.* Plans composed of objects, test and input-output specifications are called *temporal plans.*

Input-Output Specifications

An example of an input-output specification is shown at top of Fig. 4-1. An input-output specification is drawn as a solid rectangular box with solid arrows entering at the top and leaving the bottom. Each arrow entering at the top represents an input; each arrow leaving the bottom represents an output. Each input and output has a role name.[1] For example, the input-output specification depicted in Fig. 4-1, **Newterm**, has three inputs, named Old, Arg and Input; and one output, named New.

Input-output specifications also have *preconditions* and *postconditions.* The preconditions involve only the inputs; the postconditions involve both the inputs and the outputs. The simplest kind of such conditions are restrictions on the type of each role individually. These are usually indicated in plan diagrams in parentheses after the role name. For example, in Fig. 4-1 we see that Newterm.Old is expected to be a sequence; and that Newterm.Arg is expected to be a natural number. **Object** as a type restriction, as for Newterm.Input, means that there is no more specific restriction on the given role.

In this chapter and the following three, constraints *between* the inputs and outputs of an input-output specification will be described informally in English, as they are relevant to the current discussion. For example, all the terms of Newterm.New are constrained to be identical to the corresponding terms of Newterm.Old, except for the Newterm.Arg*th* term which is equal to Newterm.Input. The interested reader may refer to the appendix for a formal statement of the preconditions and postconditions of any particular input-output specification (use index to find page number). These constraints are written in a standard logical language defined in Chapter Eight.

1. In this chapter input-output specifications are primitive. In Chapter Eight, however, input-output specification specifications are treated formally as composite plans whose parts are objects and situations. This is why the inputs and outputs are roles.

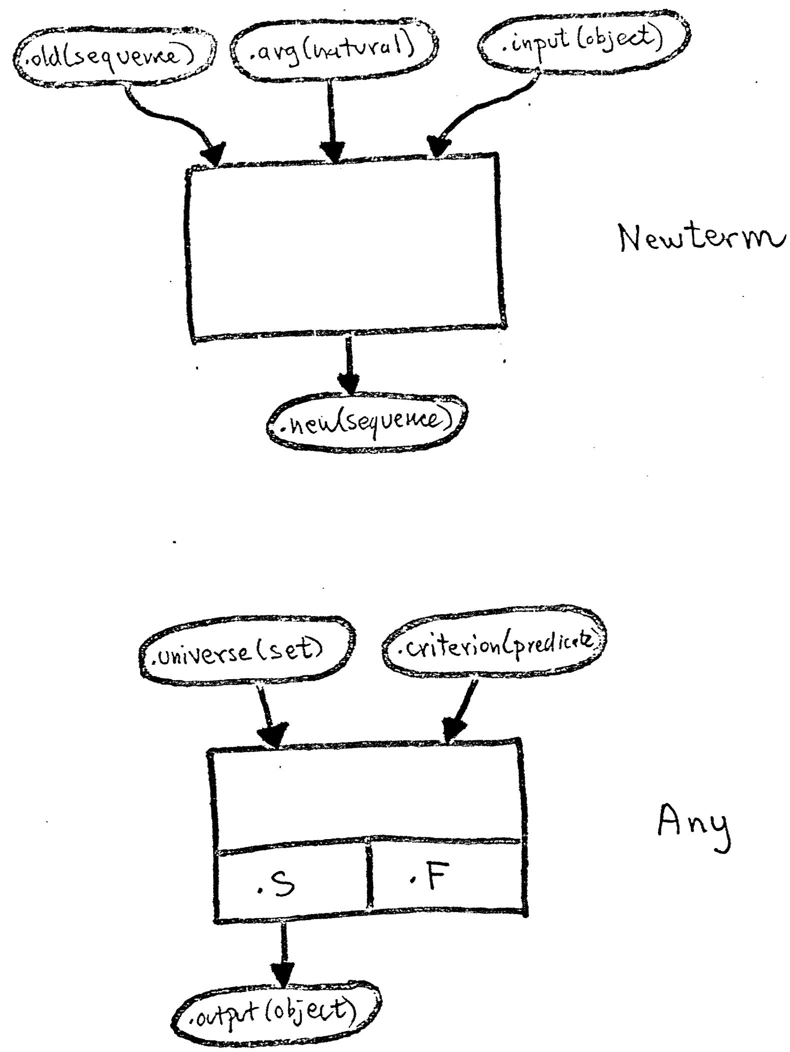


Figure 4-1. An Input-Output Specification and a Test Specification.

To reduce the clutter in more complicated plan diagrams later in this document, some information will be omitted when it can easily be inferred by the reader. For example, type restrictions (especially Object) will often be omitted for input-output specifications which should be familiar by that point in the discussion. Input and output role names will also sometimes be omitted, in which case the same left-to-right order used when the specification was first defined (which is also listed in the appendix) is to be assumed.

Test Specifications

A test specification is drawn as a solid rectangular box with a divided bottom section, as shown in the lower part of Fig. 4-1. The inputs and outputs of a test specification are notated in the same way as the inputs and outputs of an input-output specification. For example, the test shown in Fig. 4-1, has two inputs, named Universe (a set) and Criterion (a predicate), and one output named Output (an object). A test also has preconditions and postconditions, just like an input-output specification.

A test specification differs from an input-output specification in that two distinct output situations are specified. Which one occurs depends on whether or not a given relation (called the *condition* of the test) holds true between the inputs. If the test condition is true, then the test is said to *succeed* and the outputs indicated on the "S" side of the box are available; otherwise the test is said to *fail*, and the outputs indicated on the "F" side of the box are available. For example, the test specification **Any** shown in Fig. 4-1 succeeds if there exists a member of Any.Universe which satisfies Any.Criterion, in which case Any.Output is such an object; otherwise it fails and there is no output.[1]

More complicated tests with more than two cases can be represented by composing binary tests. Alternatively, the test notation is generalizable to more than two cases.

As with input-output specifications, the preconditions, postconditions and test conditions of test specifications in the following three chapters will be described informally in English in the text and formally in the appendix.

Control Flow

Fig. 4-2 shows how control flow arcs (hatched arrows) are used to connect input-output and test specifications to specify conditional behavior. This plan, called **Cond**, is the basic "if-then-else" construct in the plan calculus. Cond.If is restricted to be an instance of **Test**, which is the minimal test specification, i.e. all other test specifications are extensions of Test. Cond.Then and Cond.Else are restricted to be instances of **In+Out**, which is the minimal input-output specification. Note that the definition of In+out allows a degenerate action of doing nothing, so that conditionals with only one branch may be represented.

1. Note that at this level of abstraction, no commitment is made as to whether or not this test modifies its inputs. This constraint is added when necessary in a plan in which an Any test is used.

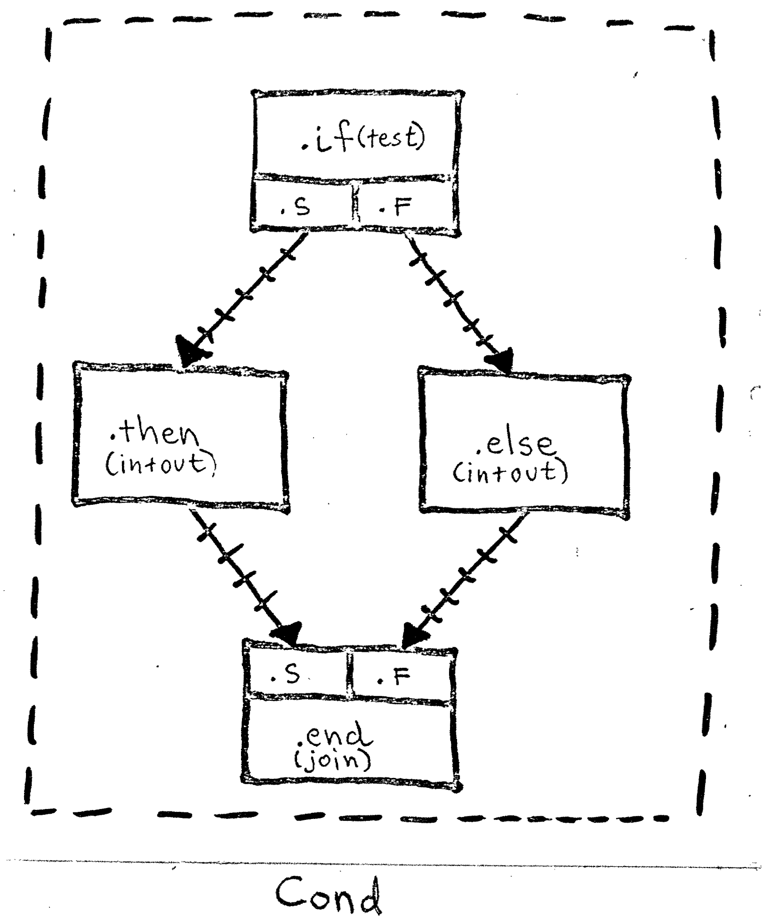


Figure 4-2. A Conditional Plan.

The End role of Cond introduces a third primitive closely related to input-output and test specification, namely *join* specifications.[1] Joins are the mirror image of tests. A join specification is drawn as a solid rectangular box with the top part divided into "S" and "F" parts, corresponding to the succeed and fail cases of the matching test. Unlike tests, however, joins do not represent any real computation. Joins are a technical artifact used to rejoin the two branches of a conditional block, as in Cond. **Join** is the minimal join specification.

An extension of Join, called **Join-output**, will be shown later. In addition to joining control flow, Join-output has input and output roles which specify the connection between which branch of a conditional is executed and which of two possible inputs is made available for further computation. For example, in the following code the input to C comes either from A or from B depending on the test P.

```
(C (COND ((P ...) (A ...))
         (T (B ...))))
```

Data Flow

Intuitively, data flow specifies equality between two data roles in a temporal plan, especially between the output of one input-output specification or test and the input of another. Data flow is indicated in plan diagrams by solid arrows, as shown in Fig. 4-3.

Fig. 4-3 shows the plan for the standard implementation of a membership test (Member?) on a set implemented as a discrimination function. This plan has two roles: Discriminate and If. The Discriminate role is restricted to be an instance of @Discrimination. (@Discrimination is a specialization of @Function in which the Op is a discrimination function and the Output is therefore a set.) The If role is restricted to be an instance of Member?, which tests whether the Input is a member of the Universe set.

The data flow arc between Discriminate.Output and If.Universe in the plan of Fig. 4-3 means that the Universe of the test is the same as output of the Discriminate operation. This data flow arc does *not* mean, however, that the test must immediately follow the Discriminate operation. An arbitrary amount of computation may occur between the end of the Discriminate operation and the beginning of the test, as long as the set involved is the same at the time the test begins as when the Discriminate operation ended.

Temporal Plans

Fig. 4-3 is an example of a temporal plan. Such plans in general have roles which are input-output, test and join specifications with data flow and control flow constraints between them. Temporal plans are drawn with a dashed box enclosing the boxes which define the roles. A very natural way of understanding the meaning of such diagrams in terms of the propagation of data and control tokens through the acyclic[2] directed graph of data and control arcs. This model is essentially the one used in data flow schemas [19].

1. Joins were introduced into the plan calculus by Waters [73].
2. Loops are represented as tail recursions.

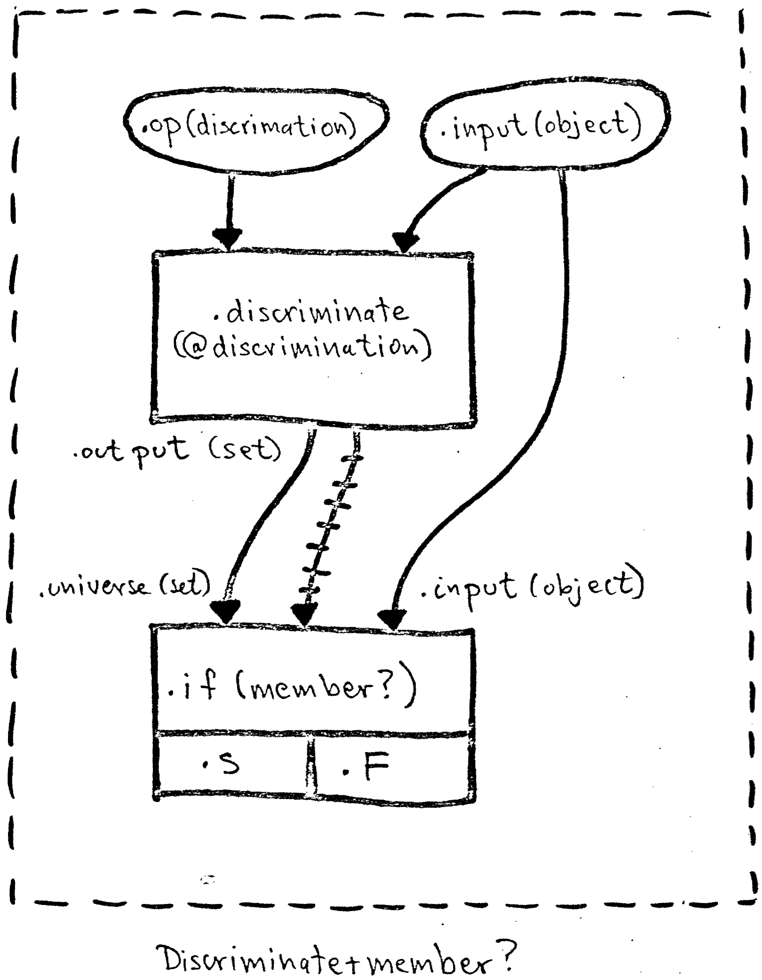


Figure 4-3. A Temporal Plan With Data Flow.

In the token propagation model of temporal plans, control flow arcs are treated no differently than data flow arcs. When an input-output box has received tokens on all of its incoming arcs, it is "activated" and generates tokens with the appropriate properties (according to its input-output specifications) on all of its outgoing arcs.[1] If an output goes to the inputs of several other boxes (i.e. an arc splits along its way into two or more arcs), then tokens passing over that arc are duplicated the appropriate number of times so that the same object is available at each input location. Control flow tokens have no properties; their only function is to enable activitation.

A test box is activated the same way as an input-output box, i.e. when it has received tokens on all of its incoming arcs. It then generates tokens either on all of the arcs leaving the success side of the box, or on all those leaving the failure side, depending on the properties of the incoming objects. A join has the complementary behavior. It is not activated until it has received all the tokens on one or the other input side. It then generates all its output tokens with properties according to its specifications (since joins involve no computation, the output tokens are always the identical to the input tokens).

Data Plans

Data plans are plans whose roles are restricted to primitive data objects or other data plans. Data plans are drawn as dashed ovals. Primitive data objects are drawn as solid ovals. For example, the data plan **Segment**, shown in Fig. 4-4, has three roles named Base, Upper and Lower, restricted to be a sequence and two natural numbers, respectively. The constraints between roles are that the Upper and Lower numbers are each less than or equal to the length of the Base sequence, and that the Lower number is less than or equal to the Upper number. (Again, these constraints are written formally in a logical language, the details of which are being suppressed until Chapter Eight.)

Recursive Plans

Recursion in plan diagrams is indicated by a spiral line as shown in Fig. 4-5. The minimal singly recursive plan is called Single-recursion. It has only one role, Tail, which is constrained to be an instance of itself. All other singly recursive plans are extensions of Single-recursion.

For example, the singly recursive plan in Fig. 4-5, called **Iterative-Generation**, describes a part of a loop in which on each iteration some function (Action.Op) is applied to an input, with the resulting output becoming the input to the application (Tail.Action) of the same function on the next iteration. The following code fragment suggests such a computation in which the Action is CDR.

```
(PROG (L)
   LP ...
      (SETQ L (CDR L))
      (GO LP))
```

1. Thus all input-output specifications require termination.

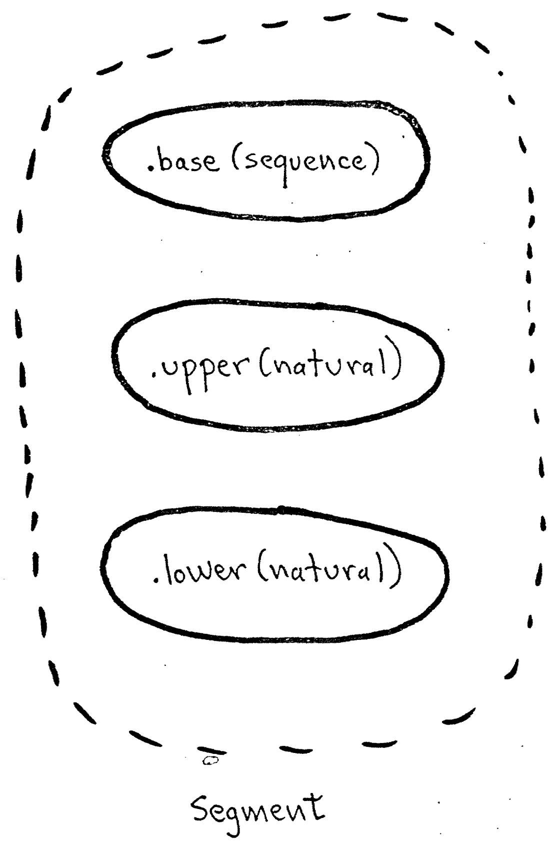


Figure 4-4. A Data Plan.

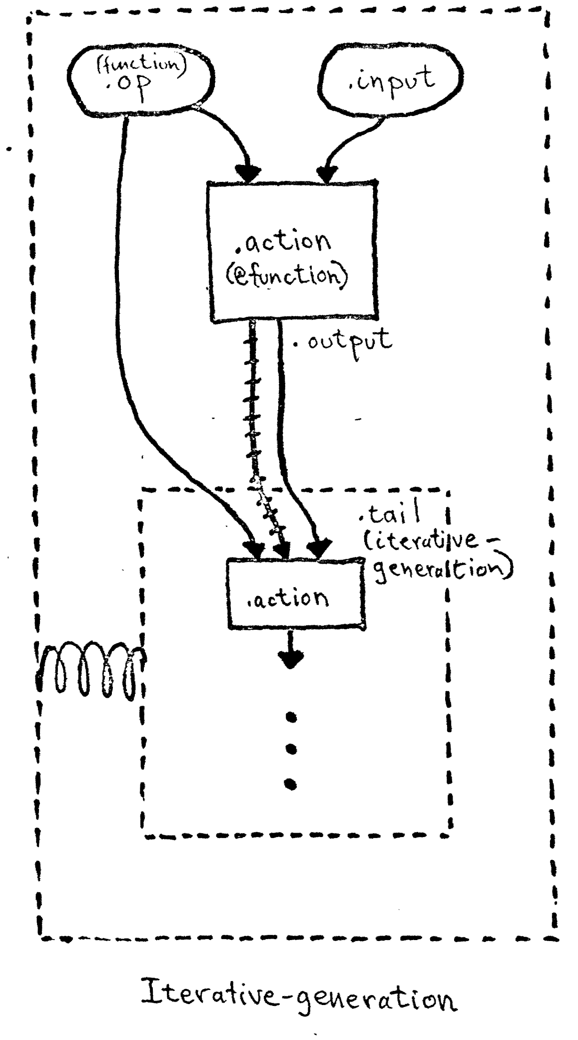


Figure 4-5. A Recursive Plan.

4.3 Surface Plans

In conventional programming languages, such as Lisp, Fortran or PL/1, it is possible to construct many different programs which, from the point of view of the plan calculus, specify the same computations. Difference in the names of variables is the most trivial example of this kind of uninteresting variability. Most programming languages also provide many different mechanisms for achieving the flow of data from one operation to another. For example, in Lisp we could write either

```
(SETQ X (F ...))
...
(G X)
```

or

```
(G (F ...))  .
```

Similarly, the following two constructions specify essentially the same control flow.

```
(PROG (...)
      ...
   LP (COND (P (RETURN NIL)))
      ...
      (GO LP))

(PROG (...)
      ...
   LP (COND (P)
            (T ...
               (GO LP))))
```

Combining all three of these kinds of superficial variation, we can construct the following two versions of the code for BUCKET-RETRIEVE (the first version is from the scenario), which illustrate how different the same program can appear. Part of the advantage of the plan calculus over programming languages for our purposes is that both of these versions translate to the same surface plan (shown in Chapter Five).

```
(DEFINE BUCKET-RETRIEVE
   (LAMBDA (BUCKET INPUT)
      (PROG (OUTPUT)
         LP (COND ((NULL BUCKET)(RETURN NIL)))
            (SETQ OUTPUT (CAR BUCKET))
            (COND ((EQUAL (CAR OUTPUT) INPUT)
                   (RETURN OUTPUT)))
            (SETQ BUCKET (CDR BUCKET))
            (GO LP))))

(DEFINE BUCKET-RETRIEVE
   (LAMBDA (BKT KEY)
      (PROG (ENTRY)
         LP (COND ((NULL BKT))
                  ((EQUAL (CAR (SETQ ENTRY (CAR BKT))) KEY)
                   (RETURN ENTRY))
                  (T (SETQ BKT (CDR BKT))
                     (GO LP))))))
```

From the standpoint of program analysis, a surface plan can be thought of as an abstraction of the data flow and control flow in a program, without abstracting the primitive data structures and operations. From the standpoint of program synthesis, a surface plan is the lowest level representation of the program design, which is then translated to code in a standard programming language.

Programming Language Semantics

In order to translate between a given programming language and surface plans,[1] the primitives of the programming language are divided into two categories: connectives, such as PROG, COND, SETQ, GO and RETURN in Lisp, which are concerned solely with implementing data and control flow; and the objects, relations, and actions of the language, such as numbers, dotted pairs, arithmetic relations, CAR, CDR and CONS. The first category of primitives is translated into the pattern of control and data flow arcs (including tests and joins) between other specifications defined in terms of the second category of primitives.

The translation of the second category of primitives (i.e. non-connectives) into the plan calculus is done in three steps, each of which involves some judgement. The first step is to identify a set of basic object types in the language. For example, Lisp can be viewed as having four basic types of objects: atoms, dotted pairs, vectors, and integers.[2]

The next step is to choose an appropriate set of primitive relationships between objects. For example, there are two primitive functions on dotted pairs, **Car** and **Cdr**, with functionalities as shown below. (**Datum** is the union type of atoms, dotted pairs, vectors and integers.)

Cdr: dotted-pair → datum
Car: dotted-pair → datum

Note that the Car and Cdr functions above are *not* the same as the CAR and CDR operations of the Lisp programming language, but are the vocabulary in terms of which the effect of these and the other builtin Lisp operations will be specified. Due to the presence of side effects in Lisp, it is important to distinguish carefully between the notion of a relationship like Car, which holds between two objects at a given point in time, and an operation, like the application of CAR, which has an input and an output, which are in the Car relation to each other.

The final step in translating from Lisp to surface plans is to translate the primitives operations such as CAR, CDR, CONS, RPLACA and RPLACD, into input-output specifications in terms of the primitive relations, such as Car and Cdr. For example, CONS becomes a input-output specification which takes as input two data objects, and returns as output a dotted pair whose Car and Cdr are the first and second inputs, respectively. RPLACA and RPLACD become input-output specification which modify the Car and Cdr functions (i.e. specializations of Newarg).

1. This has been implemented for Lisp by Waters [74].
2. This is the mathematical notion of an integer. The distinction between this and the fixed width computer representation of an integer in Lisp is not made here, because there are no plans in the current library which require this distinction.

Two additional primitive relations in Lisp are **Null** and **Eq**, with functionalities as shown below.

> **Null**: datum → boolean
> **Eq**: datum × datum → boolean

Similarly, the distinction is made between a relation and a computation which tests whether that relation holds for a given tuple of objects. For example, code such as the following constructions with `COND` is translated into the plan calculus as test specifications (specializations of @Predicate) involving Null and Eq, respectively.

```
(COND ((NULL ...) ...))
(COND ((EQ ...) ...))
```

Two more primitive functions used to model Lisp in the plan calculus are the following functions on Lisp vectors (one dimensional arrays).

> **Dim**: vector → integer
> **Element**: vector × integer → datum

The primitive vector creation (`ARRAY`) and accessing (`ARRAYFETCH` and `ARRAYSTORE`) actions of Lisp are specified in surface plans in terms of these functions.

4.4 Overlays

An overlay is essentially a triple consisting of two plans and a set of correspondences between roles of the two plans. An overlay can also be thought formally as a mapping from the set of computations (or data structures) specified by one plan to the set specified by the other. For example, the following overlay,[1]

> **Composed>function**: composed-functions → function

is a mapping from instances of Composed-functions to instances of Function. **Composed-functions** is a data plan whose two roles, named One and Two, are functions with the constraint that the range of function One is a subset of the domain of function Two. Given an instance of Composed-functions, the definition of Composed>function (which is written out formally in the appendix) specifies how to view it as the implementation of a single function from the domain of function One to the range of function Two. This overlay is a many-to-one mapping, since there are many ways a given function may be implemented as the composition of two functions. Other overlays, such as between List and Sequence, are one-to-one, which amounts to an isomorphism between the two sets of instances.

An important property of overlays is that an overlay and its inverse mapping must both be *total* on the specified domain and range. This means that, given any instance of the domain type, there exists a corresponding instance of the range type. For example, using the overlay Composed>function in program analysis, if we recognize an instance of Composed-functions, it is important to know that there

1. The character ">" is intended to be read as "as".

exists a corresponding instance of Function which it implements. Conversely, for program synthesis it is important to know that for every instance of the range type of an overlay, there exists an instance of the domain type which is a valid implementation of it.

Fig. 4-6 shows the kind of diagram which is used to represent an overlay between two temporal plans. This overlay expresses how to view the composed application of two compatible functions as the application of a composed function. An overlay diagram is divided in half by a line down the middle. The left side shows the plan diagram for the domain of the overlay; the right hand side shows the plan diagram for the range. Correspondences are drawn as lines with hooks on the ends which connect roles on one side with roles on the other.

The domain of the overlay in Fig. 4-6 is **Composed-@functions**, which has three roles: One and Two are instances of @Function, and Composite is an instance of Composed-functions. Data flow constraints in the Composed-@functions plan are such that the functions Composite.One and Composite.Two become the inputs One.Op and Two.Op, respectively; and One.Output becomes Two.Input. The range of the overlay is @Function.

Correspondences in overlay diagrams are both labelled and unlabelled. Unlabelled correspondences denote equality between the indicated roles. Labelled correspondences indicate equality between the value of labelling function applied to the role on the left and the role on the right. The function involved in such correspondences is most often another overlay.

For example, there are three correspondences in Fig. 4-6. The topmost correspondence says that the Composite role of Composed-@functions on the left hand side (an instance of Composed-functions), viewed as a function according to the overlay Composed>function, is equal to the Op role of @Function on the right. Note that the overlay Composed>function, defined earlier, is being used here to define a larger overlay which includes composed functions. This will occur twice more later in this section.

The other two correspondences in Fig. 4-6 are simple equalities. The first correspondence means that for an instance of Composed-@functions and an instance of @Function related as Composed>@function, Composed-@functions.One.Input is equal to the object filling @Function.Input. Similarly Composed-@functions.Two.Output is equal to @Function.Output

The reader may note that in the formal definition of Composed>@Function in the appendix there are two more correspondences which are not shown in Fig. 4-6: the input situation of Composed-@functions.One is identified with the input situation of @Function; and the output situation of Composed-@functions.Two is identified with the output situation of @Function. To reduce clutter, such correspondences between input and output situations will usually omitted in overlay diagrams when they can be naturally inferred.

Fig. 4-7 shows another overlay involving composed functions. This overlay, **Newvalue-composite>newvalue**, expresses the idea that, given a function implemented as a composition, a Newvalue operation on component Two of the composition can be viewed as a Newvalue operation on the whole function. This overlay is used in the analysis of the symbol table add and delete programs of Chapter Two. The hash table in those programs is viewed as a function implemented as the composition of two functions: a numerical hash function which doesn't change, and a sequence (implemented as an array), which is modified to insert new entries.

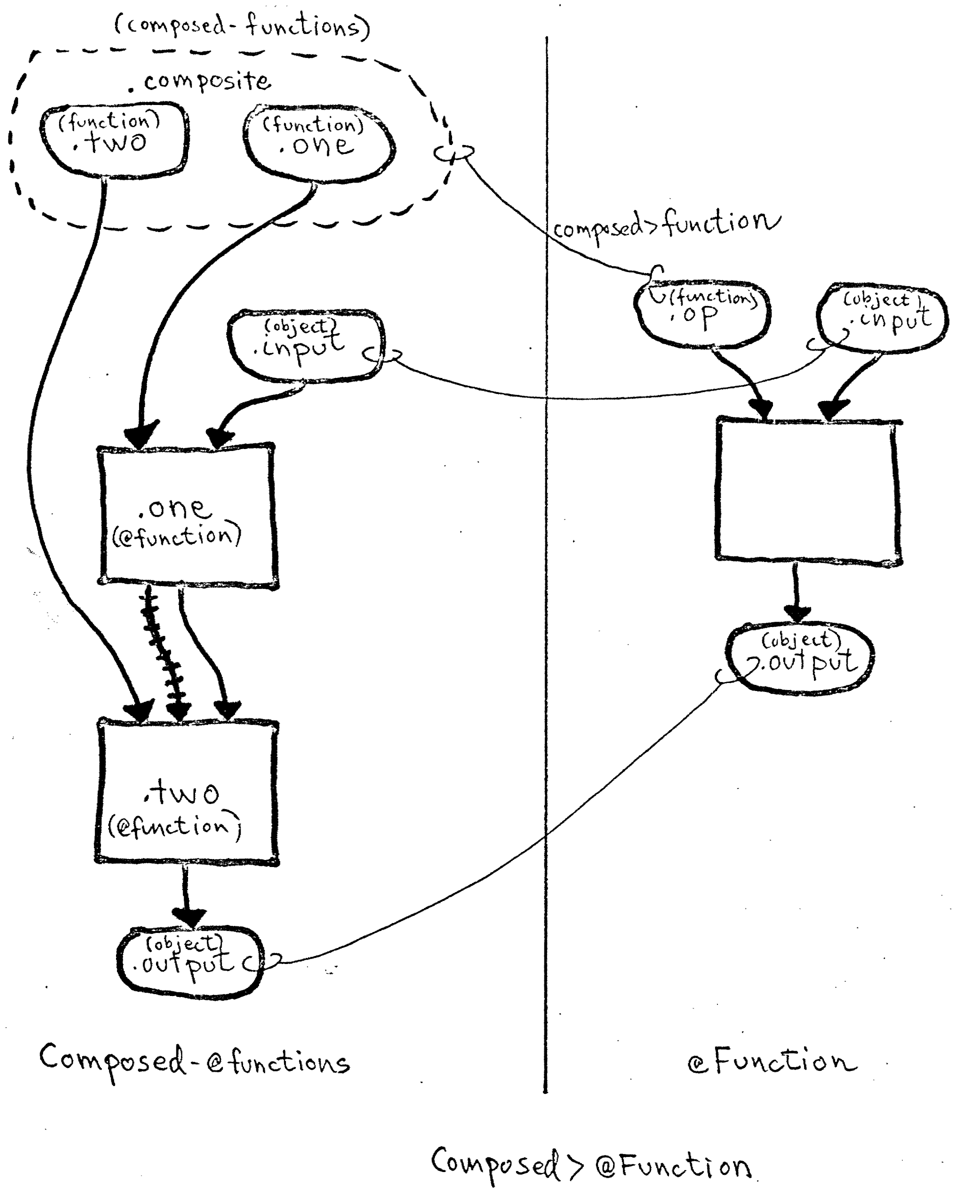


Figure 4-6. Applying a Functional Composition.

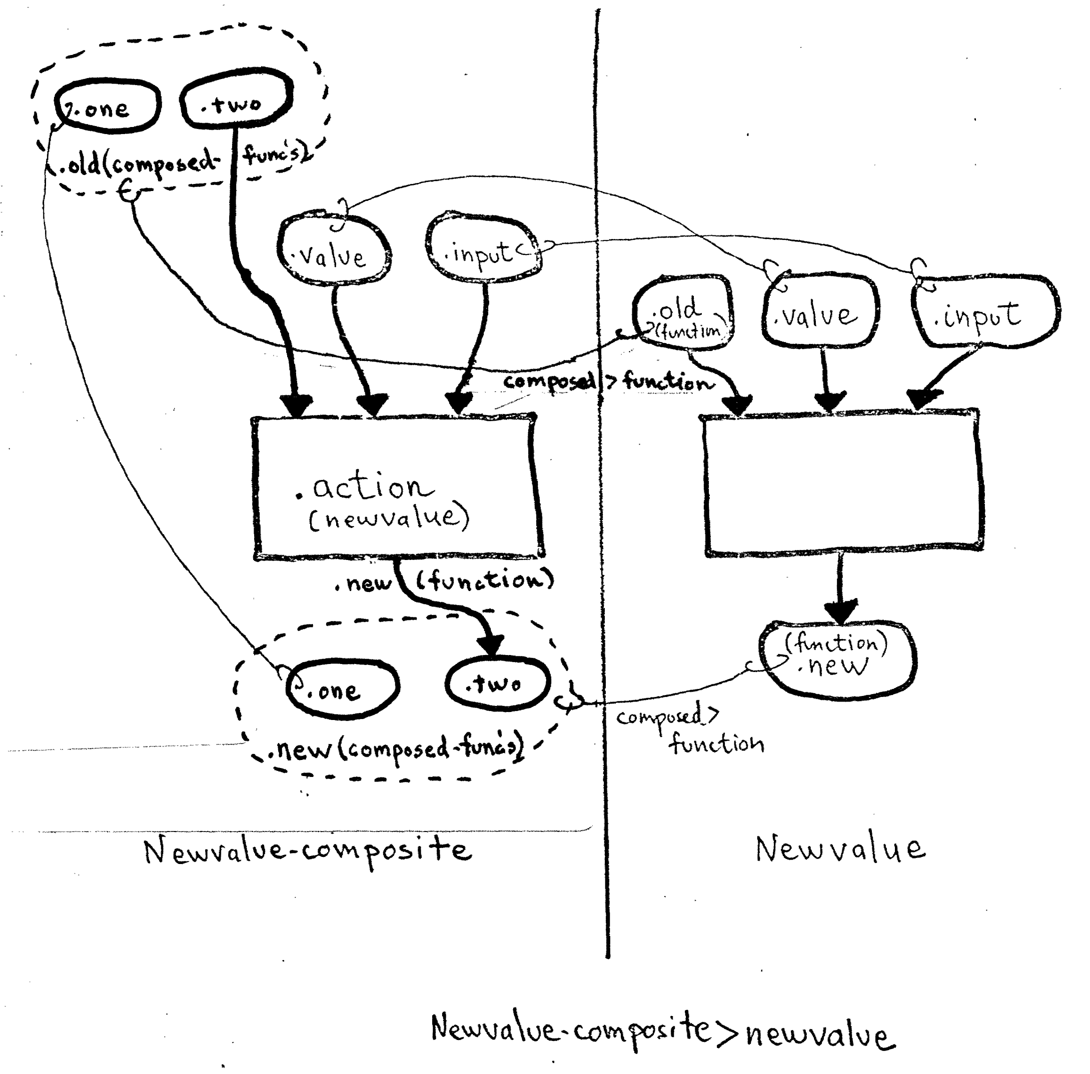


Figure 4-7. Implementing Newvalue for Composed Functions.

Notice the equality constraint between Old.One and New.One on the left hand side in Fig. 4-7. This style of building up larger plans by making use of instances of already defined plans and constraining certain components to correspond, allows us to be very concise. More important, we have separated what is novel about a particular plan, like Newvalue-composite, from what it has in common with other plans. Similarly notice that the Newvalue-composite>newvalue overlay makes use of the Composed>function overlay twice in its definition.

A Familiar Example

This section presents a second introductory example of overlays: the implementation of lists using an array and an index. This particular implementation is included here because it is a familiar example from many other papers on representing programming knowledge.

We begin with the idea of viewing a segment of a sequence between two bounds as a sequence.[1] This is formalized by the overlay **Segment>sequence**, which says (see appendix) that the terms of the implemented sequence correspond to the terms of the base sequence, offset by the lower bound.[2]

A specialization of Segment is Upper-segment, in which the upper bound is equal to the length of the base sequence. Upper-segment is a data plan often used to implement a list. The head of the implemented list corresponds to the term of the base sequence indexed by the lower bound, and the tail of the list is recursively defined as the list implemented by the upper segment which has the same base sequence with one plus the lower bound. The empty list (Nil) is implemented by a segment in which the lower bound meets the upper bound, i.e. when the lower bound is equal to the length of the sequence. This implementation is specified formally by the overlay **Upper-segment>list** in the appendix.

Fig. 4-8 defines the overlay **Bump+update>push**, which shows how to implement a Push operation on a list implemented as described above. The plan on the left hand side, Bump+update, has four roles: Bump, an instance of **@Oneminus** (the specialization of @Function when the Op is Oneminus); Update, an instance of Newterm; and Old and New, instances of Upper-segment. The essence of this plan is to update the term of the base sequence at one minus the lower bound. The correspondences in the overlay specify how this plan can be viewed as a Push operation by viewing Update.Old together with the Bump.Input as the Old input of Push (implemented according to Upper-segment>list), viewing Update.Input as the Input of Push, and viewing Update.New together with the Bump.Output as the New output of Push (again, according to Upper-segment>list).

Similarly, Fig. 4-9 defines the overlay **Fetch+Update>pop**, which specifies how to implement a Pop operation on a list implemented by Upper-segment>list. Here we see that the base sequences of the old and new upper segments are the same. One is added to the lower bound. The Output of Fetch corresponds to the Output of Pop. The Fetch and Bump operations may occur in any order since neither uses the output of the other.

1. We are skipping the step of modelling an array as a sequence, which is part of the surface plan translation.

2. This implementation "wastes" the first and and last terms of the base sequence. It can be improved by adding Oneplus and Oneminus in various places, but this would just make the example more complicated without adding any new ideas.

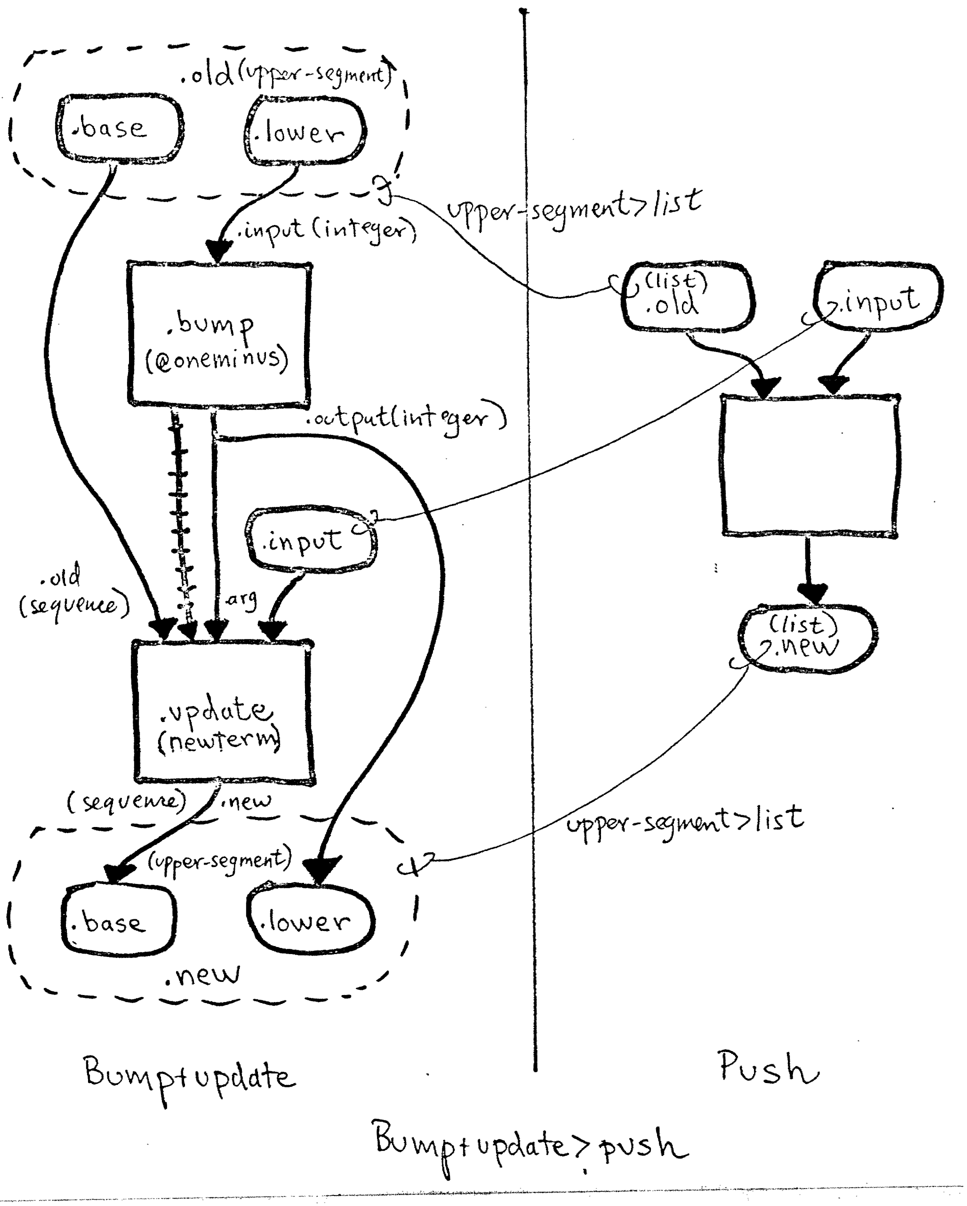


Figure 4-8. Implementation of Push.

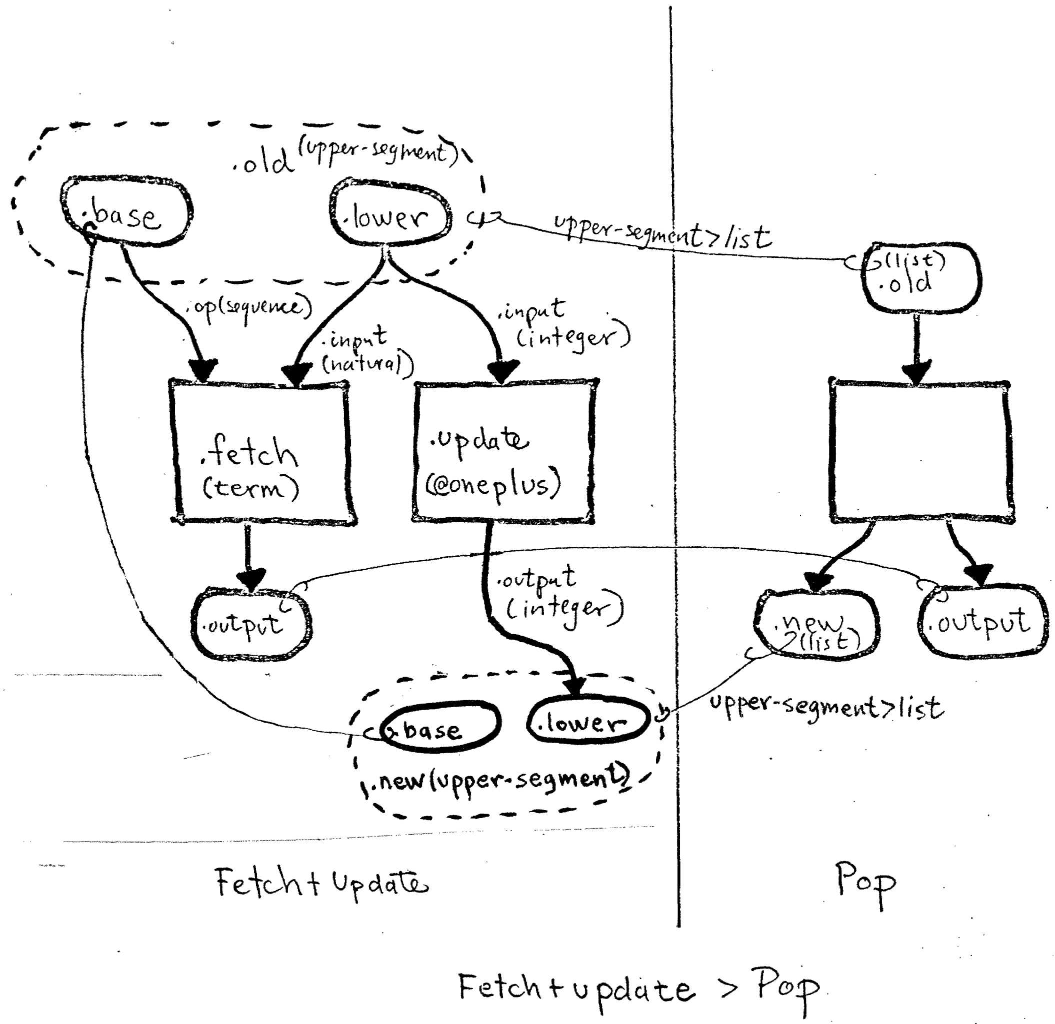


Figure 4-9. Implementation of Pop.

Using Overlays

We will see many more examples of overlays in this and the following chapters. In Chapters Five and Six we will also see how overlays are used in analysis and synthesis. For now just a few general introductory remarks are in order.

We have already seen that overlays are tool for codifying programming knowledge. An overlay can encapsulate a chunk of implementation knowledge so that it may be used many times in building up larger chunks. Such overlays express a generalization of many specific implementation strategies.

In analysis and synthesis scenarios, overlays are invoked by pattern matching against one side of the overlay and instantiating the other. For example, suppose we are in the midst of synthesizing a program and at some point we have a plan involving an instance of Push. One thing we could do is search the plan library for an overlay which has Push on the right hand side, for example Bump+update>push, and instantiate the left hand side, in this case Bump+update. The are many questions unanswered here concerning how the search and matching is performed and how the instantiated plan is hooked up with the existing plan structure. Some of these will be dealt with in Chapter Five.

In bottom-up analysis, overlays are used in a similar way to build up more abstract descriptions of the program under analysis. The first step is to recognize known plans in the surface plan translation of the program. This may involve deduction, since some of the required constraints may not yet be explicit assertions. Furthermore, this recognition process can be made more hypothesis driven by first matching against explicit assertions and then either trying to derive the rest of the required constraints, or assuming them in order to accumulate more evidence for and against the hypothetical analysis. Once a plan has been recognized, we seek to overlay it with another equivalent or more abstract plan. This is achieved by searching the library as above for overlays which have the given plan on the left hand side. Having found one, an instance of the plan on right hand side is made and add to the analysis.

Finally, overlays can be used in verification. Whether we are analyzing an existing program or have started with initial specifications for a new program to be synthesized, the final, fully verified description is a decomposition of the program into plans and sub-plans connected by overlays. From this standpoint, overlays are pre-verified lemmas in the verification of a program. Some overlays may be quite difficult to verify from first principles. However, once this has been done, they can be used over and over again. One of the goals of the library is to compile enough of these pre-verified overlays so that the verification of routine[1] programs becomes mostly a matter of combining these pieces with very little difficult deduction remaining.

1. There is an intended circularity here. I propose that what makes certain programs "routine" is that they are a straightforward combination of familiar chunks.

CHAPTER FIVE

ANALYSIS BY INSPECTION

This chapter presents a detailed scenario of the automated analysis of a program similar to part of the symbol table example of Chapter Two. The input to this analysis is the Lisp code and comments shown in Table 5-A. The output of this analysis is a hierarchy of plans which describe the computations performed by the given program at various levels of abstraction. The topmost plans in this hierarchy describe these computations in very abstract terms, i.e. in terms of set operations. The bottommost plans are very close to the code. They describe the computations in terms of the primitive data structures and operations of Lisp, such as dotted pairs, CAR and CDR. Connections between these different levels of description are represented using overlays.

The type of analysis shown in this chapter can be construed as a reconstruction of the top-down design of a program. This does not mean that the given program was actually designed that way, or that programs should be designed top-down. It only means that a top-down account is a useful way of understanding an existing program.

5.1 Why Analysis?

In a programmer's apprentice system, a complete reconstruction of the abstract structure of a program as illustrated in this chapter would seldom be required, since the intermediate levels of description would be built up incrementally as part of the development process. There are, however, other reasons for studying this type of analysis. As a practical matter, automated analysis will be useful in

Table 5-A. Lisp Code to be Analyzed.

```
; A SET OF ENTRIES IS IMPLEMENTED AS
;    A HASH TABLE ON KEYS.

; THE BUCKETS ARE IMPLEMENTED AS LISTS.

(SETQ TBL (ARRAY TBLSIZE))

(DEFINE LOOKUP
   (LAMBDA (KEY)
       (PROG (BKT ENTRY)
             (SETQ BKT (ARRAYFETCH TBL (HASH KEY)))
          LP (COND ((NULL BKT)(RETURN NIL)))
             (SETQ ENTRY (CAR BKT))
             (COND ((EQ (CAR ENTRY) KEY)
                    (RETURN ENTRY)))
             (SETQ BKT (CDR BKT))
             (GO LP))))

(DEFINE HASH
   (LAMBDA (KEY)
      (REMAINDER (MAKNUM KEY) TBLSIZE)))
```

converting from present programming technology, which deals primarily with code, to future technologies which will involve many levels of description. Furthermore, for the foreseeable future the common medium for transfer of programs between different systems is likely to be code written in a standard programming language. For both of these purposes, it is necessary to be able to reconstruct a plausible design from given code. systems.

More fundamentally, many of the capabilities required for program analysis are important in other parts of the programming process as well. For example, the ability to recognize standard computations (analysis by inspection) at various levels of abstraction is important for automating both synthesis and verification, even in an incremental system. This is because there are often several different, but equally intuitive, ways of abstracting a given computation. For example, the symbol table LOOKUP procedure can be abstracted either as associative retrieval (i.e. finding an entry in a set satisfying a given predicate), or as the application of a (partial) function from keys to entries. A programmer may be developing a program along one of these viewpoints, but the system may have to reanalyze it in a different way in order to bring the power of the plan library to bear. Furthermore, in an interactive program development system, this reanalysis need not wait until the plans involved are specific enough to be translated into code — reanalysis can be useful at all levels of abstraction.

5.2 Overview

The overall goal of the analysis described in this chapter is to decompose a given program into parts which are recognized from the plan library. This is done in four major steps. The first two steps are basically algorithmic and have been implemented. The second two steps are of a more heuristic nature, and have not yet been implemented. In summary, while this chapter gives a fairly complete account of what constitutes the analysis of a program, it only goes part way towards automating the process of constructing one.

The first step in analyzing an already written program is to translate from the given program programming language into the plan calculus. This step is viewed as a translation because it does not involve any programming knowledge other than the semantics of the programming language. The plans which are are the output of this translation step are called *surface plans.* The purpose of this translation step is to insulate the rest of the analysis process from the syntactic differences between various programming languages. Surface plans resulting from the translation of Lisp code were described briefly in Chapter Four. Code to surface plan translation has also been implemented for Fortran [73] and Cobol [24].

The second step of analysis described in this chapter is loop analysis. The purpose of this step is to decompose loops and recursions in a way which makes producer-consumer relationships explicit. Furthermore, the producer and consumer components resulting from this decomposition are often specializations of standard plans in the library. For example, temporal analysis decomposes the loop in LOOKUP roughly into three parts: CDR generation, iterative application of CAR, and iterative testing for an entry with the given key. These components are connected by data streams which represent the history of values taken on by the loop variables BKT and ENTRY. The idea for this type of loop analysis using the plan calculus was developed and has been implemented by Waters.

The final two steps of analysis in this chapter are less well worked out. The basic idea is to try to recognize known plans, first working bottom-up and then top-down. Working bottom-up entails regrouping parts of the surface plan and the temporal analysis so as to match plans in the library. One method of controlling this process is to use the *types* of the various descriptions involved (such as list, number, test, or loop) as a first filter on the grouping and matching. Also, not all plans in the library are considered in this first bottom-up matching phase. For example, with the current library, bottom-up analysis goes as far as recognizing plans which have distinctive control flow and data flow features, but does not include recognizing program structure having to do with the hash table. How far bottom-up methods can proceed with a larger plan library is an issue for further study.

The final step of plan recognition in this chapter is top-down analysis by synthesis. I assume that we are given a high level description of the program to start with. For example, for the symbol table program we are told that "a set of entries is implemented as a hash table on keys", and that "the buckets are implemented as lists". The concepts of set, hash table, key, bucket and list are all known in the current library. Furthermore, the names of the Lisp functions in Table 5-A, HASH and LOOKUP, and the names of their arguments, KEY and ENTRY, are taken as part of the program documentation indicating that these procedures implement a hashing function and associative retrieval from the set of entries, respectively.

The basic idea of analysis by synthesis is to use the plan library to generate possible implementations of the given high high level description until we find one which matches the existing bottom-up analysis. With the current library and the symbol table example, this technique appears to be feasible with simple breadth-first search through the space of possible implementations. With a larger library, some additional control mechanisms will need to be developed. Fickas [25] has done some initial work in this direction.

The approach of dividing plan recognition into a bottom-up phase and a top-down phase has the feature that programs for which the appropriate higher level plans are not in the library can still be partially analyzed at the lower levels. For example, if the methods described in this chapter, together with the current plan library, were applied to analyzing an associative retrieval data base implemented entirely with linked lists, the top-down part of recognition would fail, but we would still succeed in analyzing the structure of the program at the level of search loops and list manipulations.

The next four sections illustrate the four steps of analysis outlined above using LOOKUP. Note that there is not much to say about the analysis of the first two s-expressions in Table 5-A by themselves. These expressions simply create a Lisp vector (TBL) of a specified size and define a numerical function (HASH), both of which are used later.

5.3 Surface Plans

This section discusses the surface plan of LOOKUP in detail, explaining both the specifics of this example, and some points about surface plans in general.

The surface plan of LOOKUP is shown in Fig. 5-1 and Fig. 5-2. At the top level, this plan has three steps: application of the hashing function, fetching from the hash table, and a loop with two exits. This structure is shown in Fig. 5-1 as the plan named Lookup-surface with four roles named One, Two, Loop

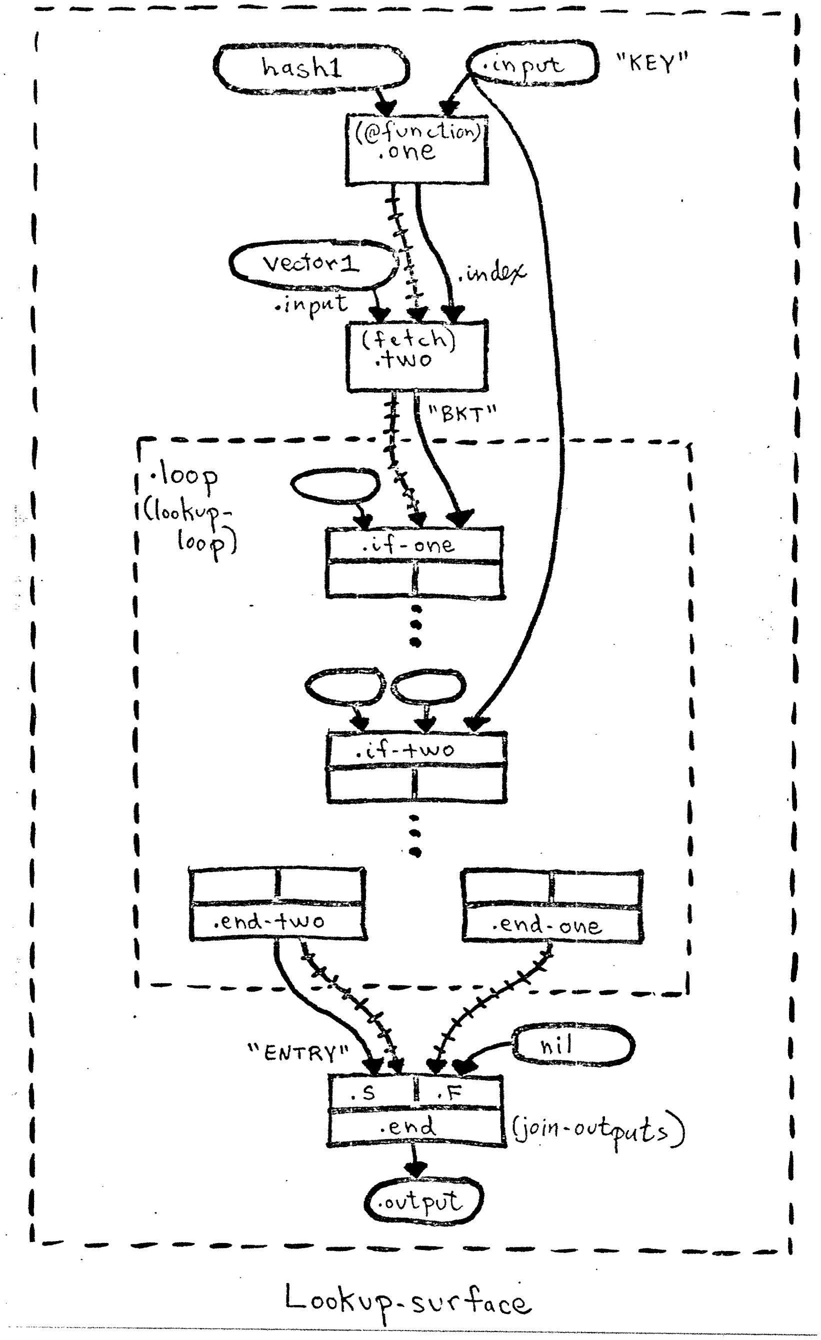


Figure 5-1. Toplevel Surface Plan for Lookup.

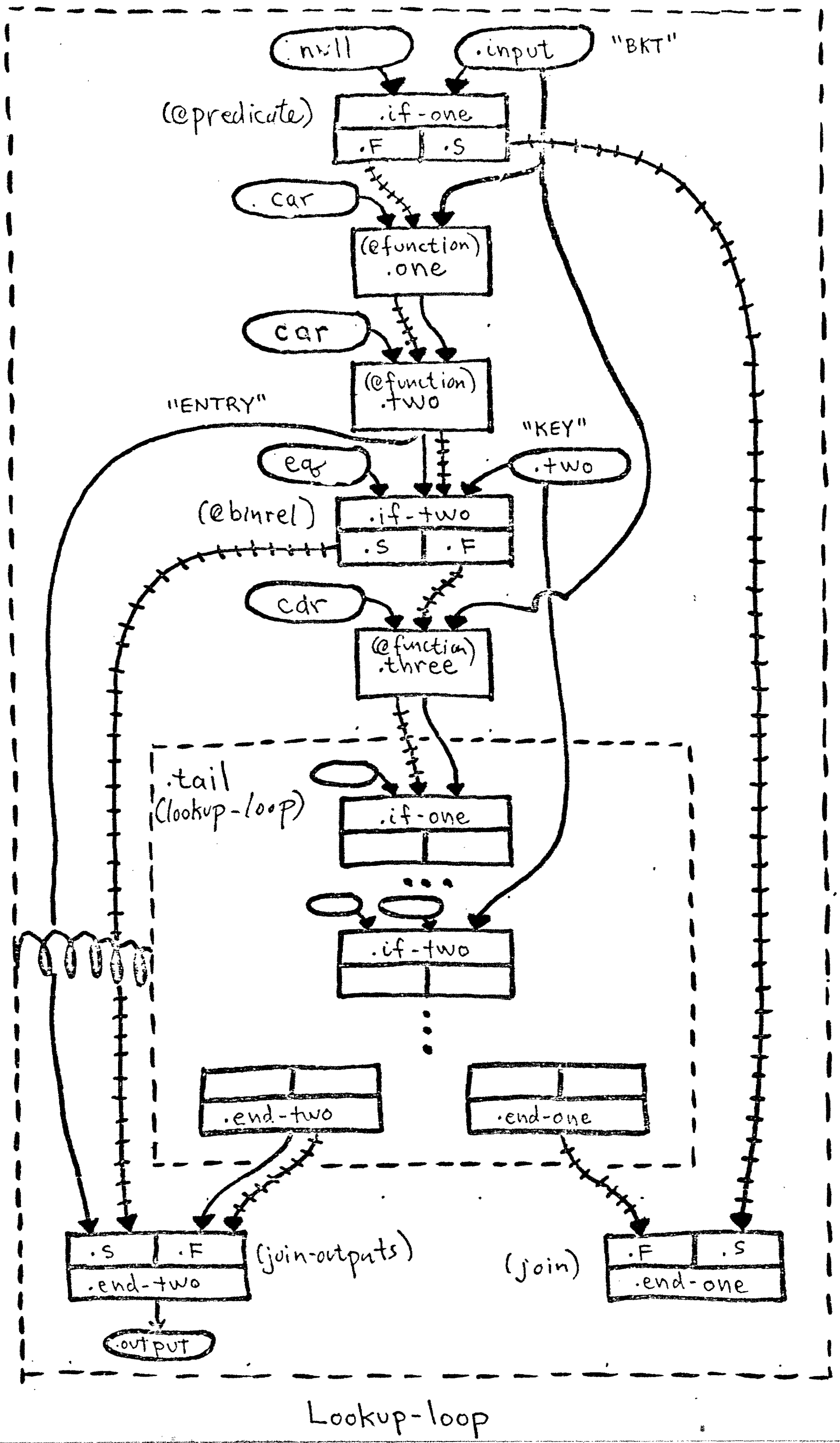


Figure 5-2. Surface Plan for Loop in Lookup.

and End. (The fourth role, End, is required to join the two cases of the loop). The Loop role of Lookup-surface is further described by another plan, Lookup-loop, which is shown partially in Fig. 5-1, and in full in Fig. 5-2. The names of these plans and their roles are generated by the translation process based on some simple conventions.

Note in these figures that inputs and outputs that are not constrained by data flow are usually either unconstrained (as far as the larger plan is concerned) or fixed to some constant. Unconstrained inputs and outputs are labelled with the appropriate role names in ovals. Roles that are fixed to constants are indicated by writing the constant inside the corresponding oval. Constants can be distinguished from role names by the absence of the point prefix. All of these notations are illustrated by Lookup-surface.One in Fig. 5-1. The function being applied (Lookup-surface.One.Op) is a constant, Hash1, which is the numerical function defined by HASH. The argument to the function (Lookup-surface.One.Input), which corresponds to the variable KEY in the code, is unconstrained. Finally, there is a data flow link between Lookup-surface.One.Output and the second input of Two.

The input-output specification of Lookup-surface.One is @Function, the application of a given function (Op) to a given domain element (Input) to compute the corresponding range element (Output). In the case of Lookup-surface.One, the function applied is Hash1.

The input-output specification of Lookup-surface.Two is **Fetch**. The inputs to a Fetch operation are, in order from left to right, Input (a Lisp vector) and Index (a valid numerical index for that vector). The Output is the indexed element of the input vector. Lookup-surface.Two.Input is constrained to be Vector1, the Lisp vector created in TBL.

After Lookup-surface.Two, control flows into Lookup-surface.Loop. As can be seen in Fig. 5-1, control exits from the loop at two different locations. These two exits correspond to the two RETURN statements in the code for LOOKUP. In one case, (RETURN ENTRY) there is also data flow out of the loop.

The surface plan for the looping part of LOOKUP is shown in full in Fig. 5-2. The most prominent feature of this plan is that it is recursively defined. In the plan calculus, loops are represented using recursive definition, as suggested by the following code.

```
(DEFINE LOOKUP
   (LAMBDA (KEY)
       (PROG (BKT ENTRY)
             (SETQ BKT (ARRAYFETCH TBL (HASH KEY)))
             (LP))))

(DEFINE LP
   (LAMBDA ()
             (COND ((NULL BKT)(RETURN NIL)))
             (SETQ ENTRY (CAR BKT))
             (COND ((EQ (CAR ENTRY) KEY)
                    (RETURN ENTRY)))
             (SETQ BKT (CDR BKT))
             (LP))))
```

This turns out to be the most convenient representation for many purposes, especially for making inductive arguments in program verification.[1] In plan diagrams, recursive definition is indicated by a curly line, as in the lower left of Fig. 5-2. This notation means that the Tail role of Lookup-loop is defined to have the same plan as Lookup-loop. Enough of the Tail is expanded in this diagram to specify the connections between one repetition of the loop and the next.

Lookup-loop has seven other roles, in addition to Tail. Three of these (One, Two and Three)[2] are applications of the primitive Lisp functions, Car and Cdr. These are the translations of `(CAR BKT)`, `(CAR ENTRY)` and `(CDR BKT)` in the code. The other four roles in Lookup-loop are various kinds of tests and joins.

Two particular kinds of test specifications used in Lookup-loop are @Predicate and @Binrel. @Predicate tests whether or not a given unary relation (Criterion) is true of given object. Similarly, @Binrel tests whether two given objects satisfy a given binary relation (Criterion).

The first test in Lookup-loop, If-one, is constrained to be a instance of @Predicate in which the Criterion is Null. If-one is the translation of the code

```
(COND ((NULL BKT)...)) .
```

When this test succeeds, control exits from the loop, as can be seen by the control flow arrow from the "S" side of If-one which bypasses the Tail. When this test fails, control passes to One and then Two, which are the translation of the following portion of the loop code.

```
(SETQ ENTRY (CAR BKT))
...(CAR ENTRY)...
```

The output of Two feeds into input One of If-two, which is an instance of @Binrel. The Criterion of this test is the primitive binary relation, Eq. If-two is the translation of the code

```
(COND ((EQ ... KEY) ...)) .
```

Note that If-two.Two (`KEY` in the code above) is unconstrained as far as the Lookup-loop plan is concerned, except that it doesn't change on successive repetitions of the loop. The fact that input Two of this test is the same as the argument to the hashing function is reflected in the constraints of Lookup-surface, as can be seen in Fig. 5-1. If this test succeeds, control exits the loop through End-two making One.Output (`ENTRY`) available outside the loop. Otherwise, Cdr is applied to If-one.Input (`BKT`), with the result feeding into the recursive invocation.

Lookup-surface.End, Lookup-loop.End-one and Lookup-loop.End-two are joins, the complementary construct to tests. There are two possible ways for control to flow into a join, and one way out.[3] Joins indicate the effect of control flow on data flow. For example, the pattern of data flow and control flow through Lookup-surface.End in Fig. 5-1 indicates that in one case the value returned by

1. Many Lisp interpreters and compilers execute tail recursive code as efficiently as code with loops in control flow, making these essentially syntactic variants.
2. As in Lookup-surface, the role names in this plan are chosen by the translation process based on some simple conventions.
3. Note that this is not a parallelism construct. In any given computation, only one or the other branch of a conditional is taken.

LOOKUP is the entry that satisfied the second exit test of the loop (the value of ENTRY); in the other case it is the constant, Nil.

5.4 Loop Analysis

The overall goal of analysis by inspection is to decompose a program into recognizable parts. In other words, we want to figure out how the surface plan for a program could be built up out of standard plans in the library. This section in particular is concerned with the analysis of singly recursive surface plans, such as Lookup-loop, which represent the looping parts of a program. It is important to note, however, that none of the analysis in this section is particular to whether a surface plan is the translation of code written in Lisp versus some other conventional programming language. Although appropriate specializations for Lisp will be emphasized for the purpose of this example, the plans and overlays introduced in this section are all quite general.

The analysis of loops takes place in two steps. In the first step, a loop is decomposed into standard recursively defined fragments. In the second step, the behavior of these fragments is abstracted in such a way that a loop can be represented by a non-recursive plan. This allows further analysis to treat the looping and non-looping parts of programs uniformly.

Loop Augmentations

The natural building blocks for non-recursive temporal plans are input-output and test specifications, which are composed using control flow and data flow. The plan library contains many standard input-output and test specifications and plans for their implementation by compositions of other input-output and test specifications. For recursively defined temporal plans, however, a different notion of composition is needed in order to make a library of standard building blocks. Loops are viewed here as being constructed by a process of *augmentation*.[1] For example, the loop of LOOKUP can be constructed starting with just the part that does the CDRing, as suggested by the following code.

```
`(PROG (BKT)
       (SETQ BKT ...)
    LP ...
       (SETQ BKT (CDR BKT))
       (GO LP))
```

This pattern of looping, in which a given function is repeatedly applied to the output of the preceding application of that function, is called **Iterative-generation.** Iterative-generation using Cdr is a common building block of many loops in Lisp. This loop can be augmented by adding the code underlined below.

1. This view of loops is taken from Waters [73]. In this reference, Waters also goes into a more lengthy justification of why a different analysis method is required for loops as compared to straight-line code.

```
(PROG (BKT ENTRY)
      (SETQ BKT ...)
   LP ...
      (SETQ ENTRY (CAR BKT))
      ...
      (SETQ BKT (CDR BKT))
      (GO LP))
```

The basic idea of augmentation is that the augmented loop does everything the unaugmented loop does, plus something extra. For example, the augmented loop above makes available in ENTRY the CAR of each successive value of BKT computed by the generation part of the loop. This pattern of augmentation is called **Iterative-application**; the function being applied in this case is Car. In general, the effect of an augmentation is to create a new sequence of data objects (such as the values of ENTRY) in the augmented loop which is related in some way to a sequence of objects (such as the values of BKT) in the unaugmented loop. Two other kinds of augmentation, which are not illustrated in the symbol table example, are filtering and accumulation. These will be discussed in Chapter Nine.

The addition of an exit test to a loop, as shown underlined below, is a kind of augmentation which is an exception to the general rule that augmentation preserves the complete behavior of the unaugmented loop, since without the exit test, the loop generates infinite sequences of data, which is not the case with the exit test present.

```
(PROG (BKT ENTRY)
      (SETQ BKT ...)
   LP (COND ((NULL BKT)(RETURN NIL)))
      (SETQ ENTRY (CAR BKT))
      ...
      (SETQ BKT (CDR BKT))
      (GO LP))
```

This kind of augmentation is called **Iterative-termination.** The reason an exit test is treated as a kind of augmentation, even though it changes the behavior of the loop, is because it effect is abstracted in the same way as other augmentations. Adding Iterative-termination creates truncated versions of the (potentially infinite) sequences which would be generated by the loop without the exit test. More than one iterative termination can be added to a loop, as shown underlined below.

```
(PROG (BKT)
      (SETQ BKT ...)
   LP (COND ((NULL BKT)(RETURN NIL)))
      (SETQ ENTRY (CAR BKT))
      (COND ((EQ (CAR ENTRY) KEY)
             (RETURN ENTRY)))
      (SETQ BKT (CDR BKT))
      (GO LP))
```

Waters has implemented a system which automatically decomposes loops according to this idea of augmentation. The basic algorithm his system uses is to iteratively remove parts of a loop which do not produce data objects required by the remaining parts. For example, for the loop example above (which is part of LOOKUP), the effect of this algorithm is to undo the augmentation steps in the reverse order they were introduced above. The plan library contains plans for many standard augmentations. The rest of this section shows some of these which are used in LOOKUP and how they are represented in the plan calculus.

The first augmentation recognized in the LOOKUP loop is shown in Fig. 5-3. On the left hand side of this figure we have the surface plan for the loop, Lookup-surface. On the right hand side is a plan from the library called **Terminated-iterative-search.** This plan captures the idea of a search loop with two exits, without specifying how the sequence of objects being searched is produced. Role If-two of this plan is a test which applies a given criterion (the same on each iteration) to the current input (provided by the rest of the loop). When this test succeeds, the current input is made available outside the loop (as End-two.Output). The other exit test (If-one) is for terminating the loop when there are no more objects in the search space.

Note that the role names of a plan in the library, such as Terminated-iterative-search, are fixed at the time the plan is catalogued. In general, role names have been chosen to have some mnemonic value relative to the given plan, but this strategy is somewhat restricted by the fact that specialized plans inherit their role names from their generalizations. For example, the most general plan for a two exit loop, of which Terminated-iterative-search is a specialization, is **Cascade-iterative-termination.** At the level of generality of Cascade-iterative-termination, it is not possible to give any better names to the two exit test roles than If-one and If-two.

The hooked lines between the left and right hand sides of Fig. 5-3 indicate how the Terminated-iterative-search plan is matched against Lookup-surface: Lookup-loop.If-one corresponds to Terminated-iterative-search.If-one; Lookup-loop.If-two corresponds to Terminated-iterative-search.If-two;[1] and there is a correspondence between the joins, End-one and End-two. The fact that the corresponding roles have the same names in this example is a coincidence. The hooked line between Lookup-loop.Tail and Terminated-iterative-search.Tail indicates that the correspondence is made recursively.

Fig. 5-3 is an example of an *overlay*. The basic idea of overlays is re-description. The plan on the left describes a set of computations — the instances of the plan. The correspondences in the figure indicate how to re-describe (part of) any such computation as an instance of the plan on the right, in this case a standard plan from the library. In order for this re-description to be possible, the constraints of the right hand plan must logically follow from the constraints of the left hand plan, substituting appropriately for the corresponding parts. It can be seen in Fig. 5-3 that this condition is met for control flow and data flow constraints (control flow is transitive).

Overlays are used to relate different levels of description in the analysis of a program. The origin of the term "overlay" is to suggest different plans being drawn on transparent slides and laying one on top of the other to line up the corresponding parts.[2] Some overlays, such as the one in Fig. 5-3, are particular to

---

1. A detail is being skipped here, which is covered in the appendix. Lookup-loop.If-two is an instance of @Binrel, test in which the binary relation Eq is applied to two inputs. Terminated-iterative-search.If-two is an instance of @Predicate, a test involving a unary relation. In order to recognize Terminated-iterative-search as indicated, an intermediate step is required in which Lookup-loop.If-two is grouped together with Lookup-loop.Two and these are viewed as the implementation of testing a composite predicate of the form `(LAMBDA (X)(EQ (CAR X) KEY))`.
2. Sussman [69] uses the term "slice" for a similar concept in the analysis of electronic circuits.

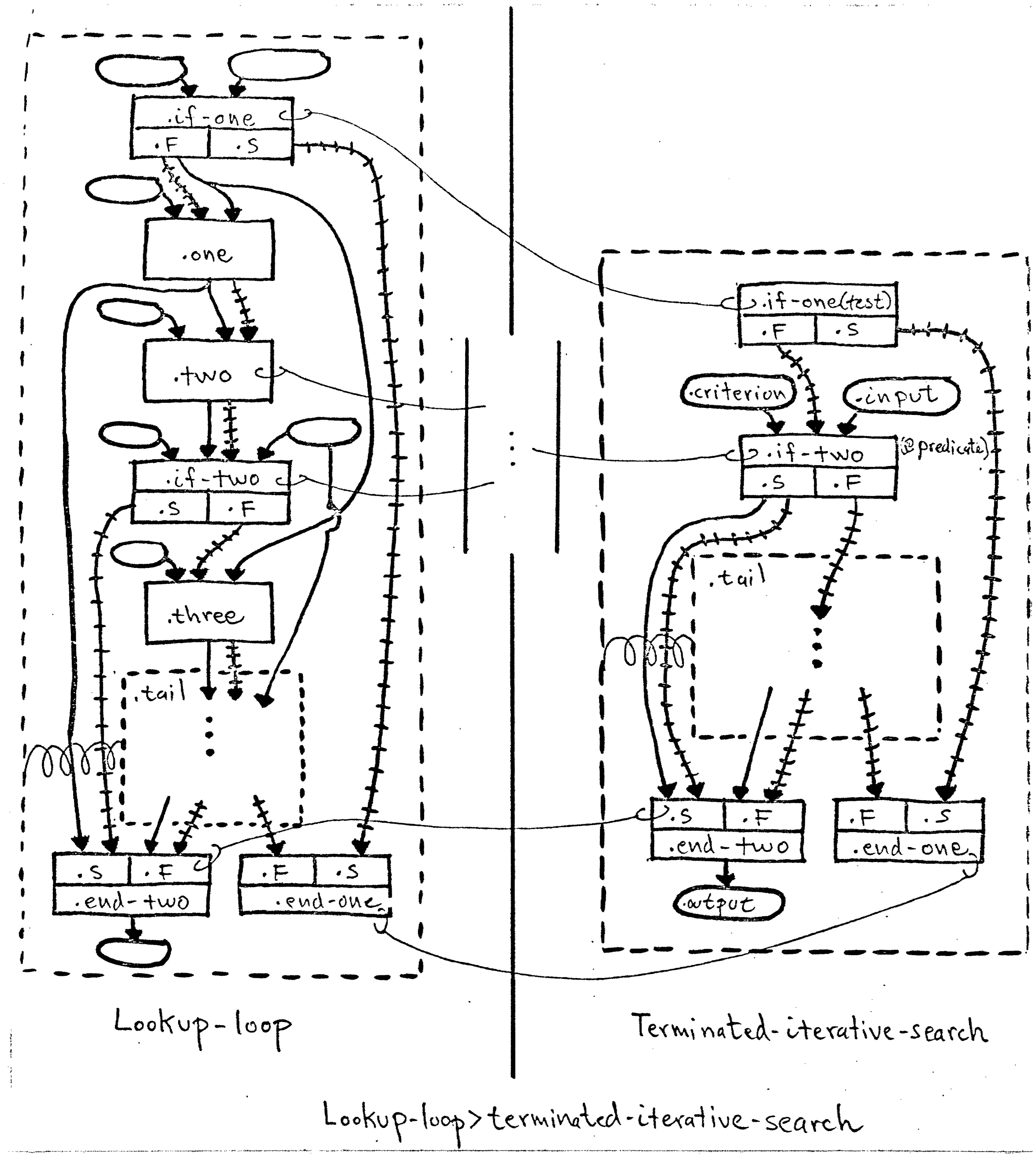


Figure 5-3. Terminated Search in LOOKUP Loop

the analysis of a specific program; others, such as those catalogued in the plan library, express re-descriptions of general applicability.

Recognition of the other augmentations in Lookup-loop takes place in a model of the loop in which the exit tests are assumed to always fail. This is called the *steady state* model. The relationship between the surface plan and the steady state model is represented using an overlay which is explained in more detail in Chapter Nine.

The first augmentation recognized in the steady state model of Lookup-loop is the iterative application of Car, shown in Fig. 5-4. On the right hand side of this overlay is the plan from the library, Iterative-application, which expresses the general idea of repeatedly applying a given function (Action.Op) to an input provided by the rest of the loop (Action.Input) to produce an output (Action.Output), which may be used by the rest of the loop. The correspondences between this plan and Lookup-loop on the left indicate that Lookup-loop.One in the steady state matches this description. Similarly, Fig. 5-5 shows the how Lookup-loop.Three is re-described as Iterative-generation.Action.

Temporal Abstraction

Given that we have decomposed a loop plan into these standard augmentations, the question remains of how to represent the connection between, say, the generation and the application parts of the loop. Temporally, the components of each computation are interleaved, but it seems more logical to view the generation and application as being composed in some way. This section shows how to construct this viewpoint.

The basic idea of temporal abstraction is to view all the objects which fill a given role in a recursively defined plan as a single data structure.[1] In terms of Lisp code, this often corresponds to having an explicit representation for the sequence of values taken on by a particular variable at a particular point in a loop. For example, in the LOOKUP loop we would like to talk about the sequence of objects iteratively generated by Cdr, i.e.

Iterative-generation.Action.Input,
  Iterative-generation.Tail.Action.Input,
    Iterative-generation.Tail.Tail.Action.Input,
      ...

which corresponds to the values of BKT at the point underlined below each time around the loop (in the steady state).

```
(PROG (BKT)
      (SETQ BKT ...)
   LP ...
      (SETQ BKT (CDR BKT))
      (GO LP))))
```

1. Both Shrobe [64] and Waters [73] use the idea of temporal abstraction, but with slightly different formalizations than presented here.

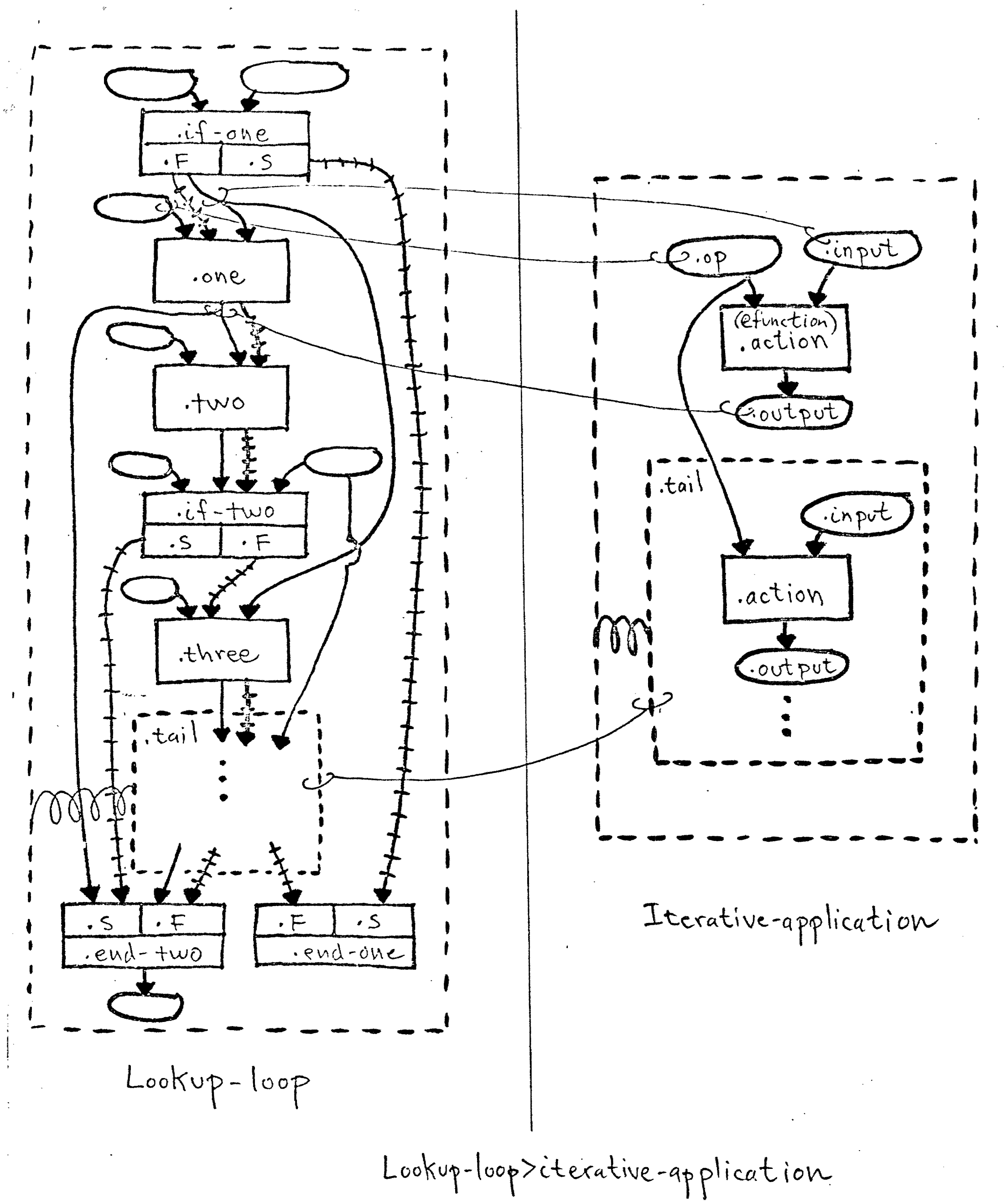


Figure 5-4. Iterative Application in LOOKUP Loop.

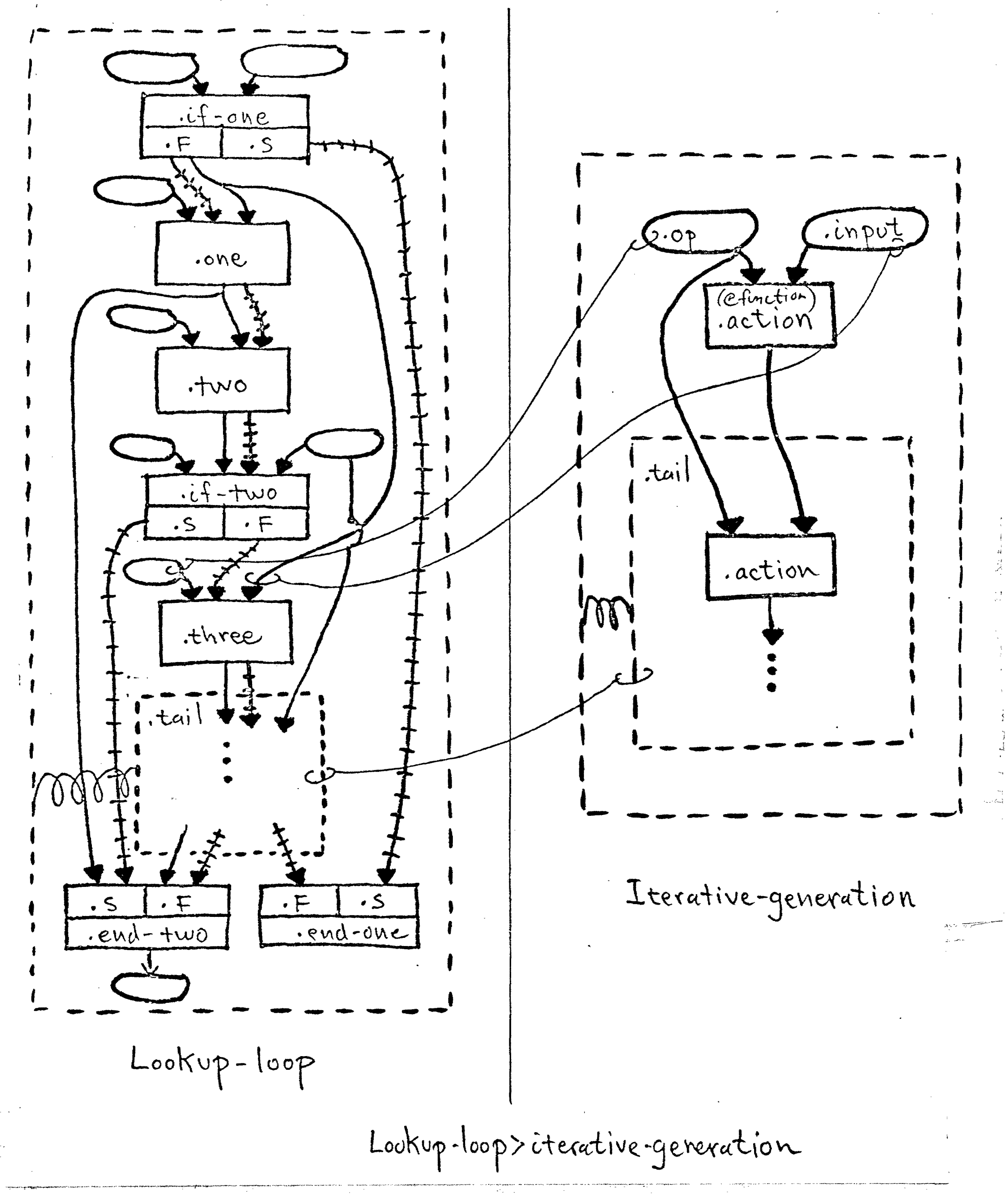


Figure 5-5. Iterative Generation in LOOKUP Loop

The bottom overlay in Fig. 5-6 shows how this abstraction is made using the input-output specification, **Iterate**, which takes as input a data structure called an iterator (which is the linear specialization of a generator), and gives as output the generated sequence. An iterator has two parts: the Seed (the starting value which will be the first term of the generated sequence) and the Op (the function which maps from one term to the next). As shown by the hooked lines in Fig. 5-6, Iterate.Input.Seed corresponds to Iterative-generation.Action.Input and Iterate.Input.Op corresponds to Iterative-generation.Action.Op. Iterate.Output represents the sequence of inputs to Action on each iteration, as described above.

Iterator is an example of a data plan — the plan for a data structure. This plan, together with Iterate and the overlays in Fig. 5-6, are part of the current library. An important feature of the plan calculus is that it allows the hierarchical description of data structures and temporal computations in a single formalism.

The top overlay in Fig. 5-6 makes the same sort of abstraction for Iterative-application. In this overlay, Iterative-application is viewed as the input-output specification, **Map**, which takes a sequence and a function (Op) as inputs, and has a sequence as output. The terms of the output sequence are the result of applying the given function to the terms of the input sequence. In the temporal view, the input sequence of Map is the abstraction of the inputs of the Action of Iterative-application on each iteration. The output sequence of Map is the abstraction of the outputs of Action. Map.Op corresponds to Action.Op. In terms of the code of LOOKUP, Map.Input represents the values of BKT at the point underlined below and Map.Output represents the values of ENTRY (in the steady state).

```
(PROG (BKT ENTRY)
      (SETQ BKT ...)
   LP ...
      (SETQ ENTRY (CAR BKT))
      ...
      (SETQ BKT (CDR BKT))
      (GO LP))))
```

Notice in this code that the value of BKT is the same at the underlined point as it is at the input to CDR. This means that in the temporal abstraction of Lookup-loop, the output sequence of the Iterate step is the same as the input sequence of the Map step.

Iterative terminations are also temporally abstracted. The effect of the NULL exit test in the LOOKUP loop is modelled in the temporal view by an input-output specification called **Cotruncate**. Cotruncate takes as input two sequences (Cotruncate.Input and Cotruncate.Co-input) and a predicate (Cotruncate.Criterion). Its output is the truncation of the second sequence at the earliest term for which the corresponding term of the first sequence satisfies the given predicate. This may sound like a somewhat obscure specification, but the idea of two parallel sequences is in fact quite basic. For example, the standard plan for computing the length of a Lisp list can be naturally viewed in terms of two parallel temporal sequences: the natural numbers, and the sequence of CDRs of the list. In the code below, the sequence of values of ENTRY at the underlined point (the output of Map) is truncated according to the Null predicate applied to the sequence of values of BKT at the underlined point (the output of Iterate).

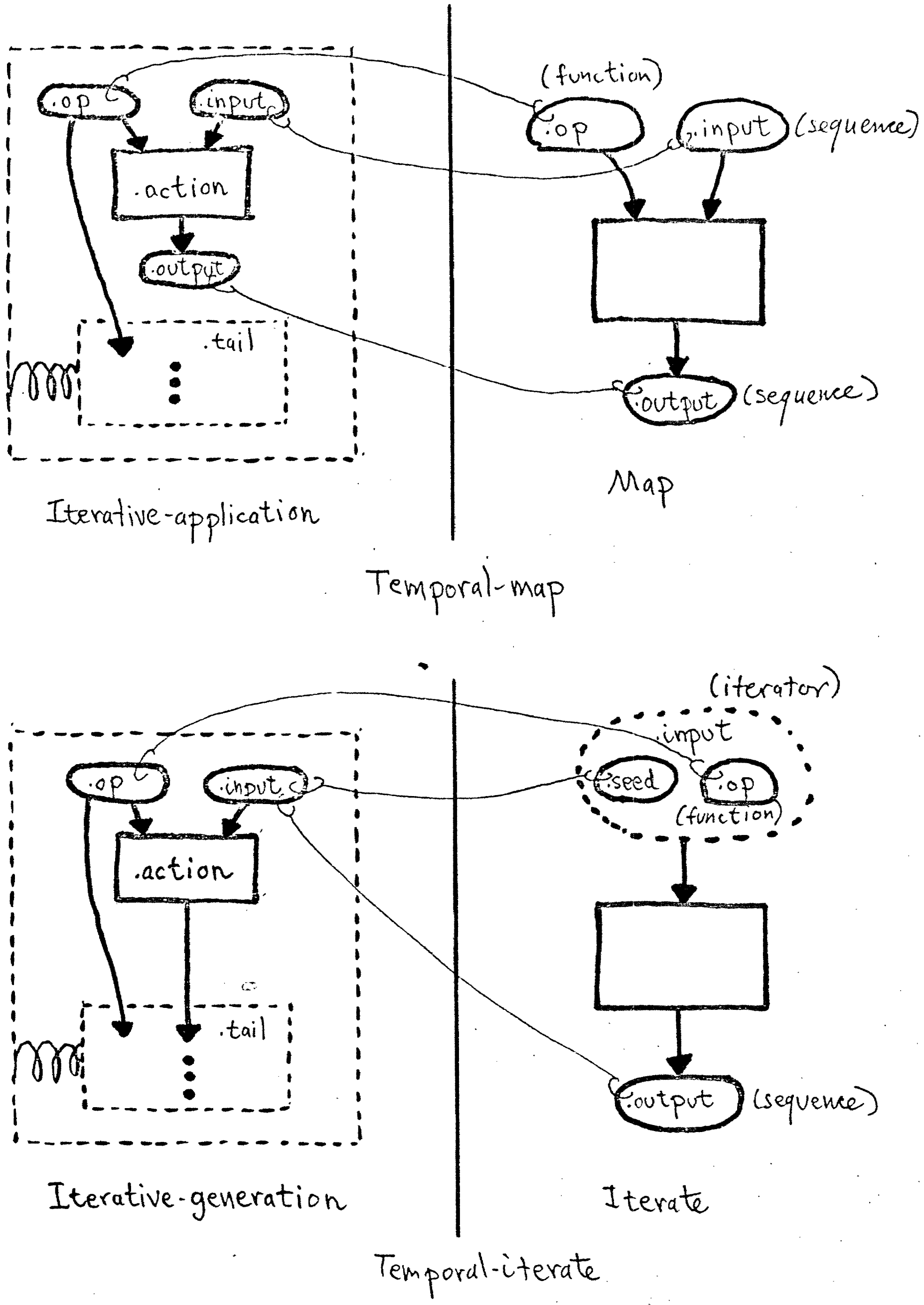


Figure 5-6. Temporal Overlays

```
(PROG (BKT ENTRY)
      (SETQ BKT ...)
   LP (COND ((NULL BKT)(RETURN NIL)))
      (SETQ ENTRY (CAR BKT))
      ...
      (SETQ BKT (CDR BKT))
      (GO LP))))
```

In the next section, we will see how this pattern of Generate, Map and Cotruncate using Car, Cdr and Null, is recognized as the standard plan for generating a list in Lisp.

The NULL exit test above has also been recognized as part of the Terminated-iterative-search plan in LOOKUP (the other exit test is EQ). Fig. 5-7 shows an overlay from the library which views Terminated-iterative-search as the temporal implementation of a standard test on sets called **Any**. Given a set (Any.Universe) and a predicate (Any.Criterion), this test succeeds if there is a member of the set which satisfies the predicate, and returns such an object (Any.output); otherwise it fails. In the temporal overlay of Terminated-iterative-search as Any shown in Fig. 5-7, Any.Universe corresponds to the set of inputs to the second exit test of the search.[1] In the code of LOOKUP, this is the set of values of ENTRY at the point underlined below.

```
(PROG (BKT ENTRY)
      (SETQ BKT ...)
   LP (COND ((NULL BKT)(RETURN NIL)))
      (SETQ ENTRY (CAR BKT))
      (COND ((EQ (CAR ENTRY) KEY)
             (RETURN ENTRY)))
      (SETQ BKT (CDR BKT))
      (GO LP))))
```

This overlay also illustrates a common form of temporal abstraction, in which we talk about the set of objects filling a given role in a recursive plan, ignoring their temporal order.[2] As we shall see, this turns out to be the appropriate level of abstraction for this example.

The relationship between the temporal abstractions of the various parts of Lookup-loop is illustrated in Fig. 5-8. This figure shows all four overlays discussed in this section applied to Lookup-loop simultaneously. In order to reduce clutter, only the data flow constraints in Lookup-loop and the correspondences which involve temporal abstraction are drawn. Notice that many of the temporal sequences on the right are the abstraction of roles of Lookup-loop which are constrained by data flow to be the same. In particular, Iterate.Output is the same sequence as Map.Input and Cotruncate.Input, and Map.Output is the same sequence as Cotruncate.Co-input.

Some of the temporal correspondences in Fig. 5-8 involve different steady state models. For example, Cotruncate.Input is the temporal abstraction of Lookup-loop.One.Output in the steady state model with no exit tests; Cotruncate.Output is the sequence which includes the effect of the Null test. This detail cannot be shown conveniently in this figure, but is explained in the next section.

---

1. More precisely, Any.Universe corresponds to the set of objects which would be searched if there were no member satisfying the predicate. This abstraction involves forming a steady state model in which exit Two always fails.
2. Formally this abstraction is done in two steps: first a temporal sequence abstraction is made; and then this ordered structure is viewed as the implementation of a set. This will be explained more fully in Chapter Nine.

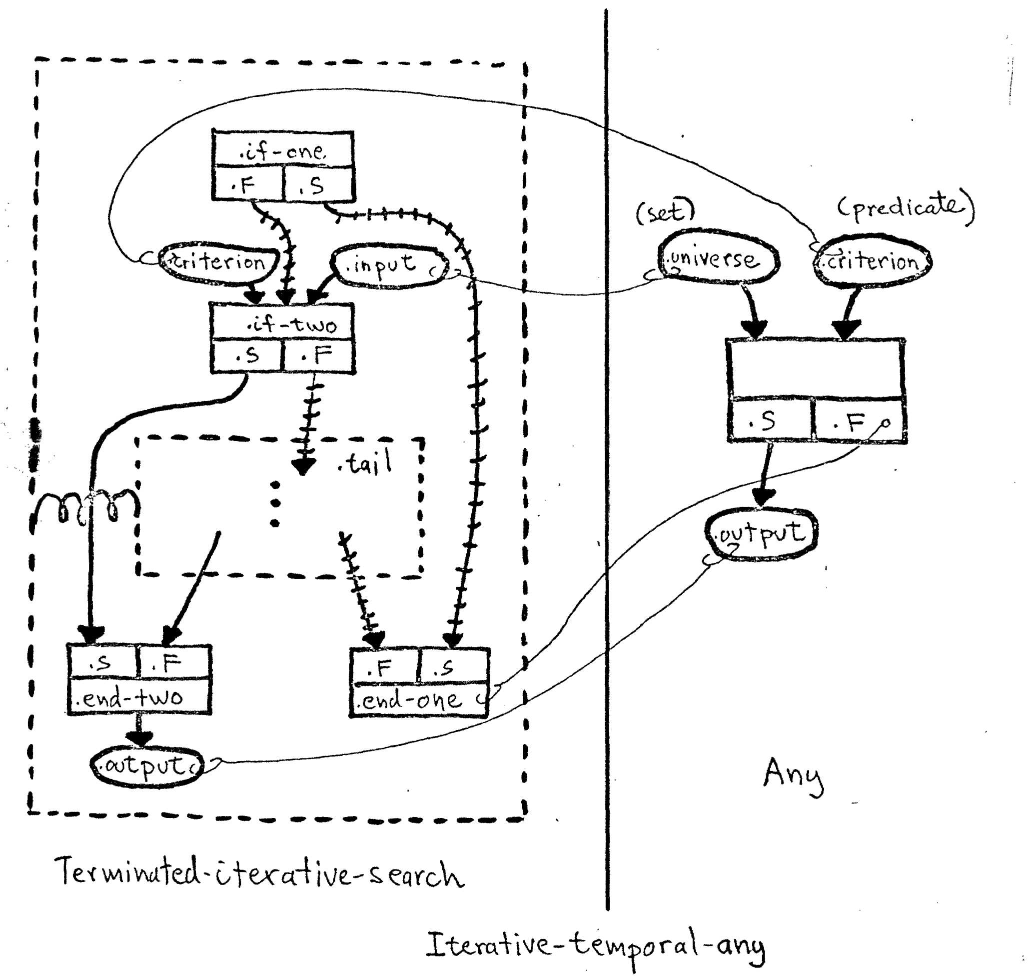


Figure 5-7. A Temporal Set Overlay.

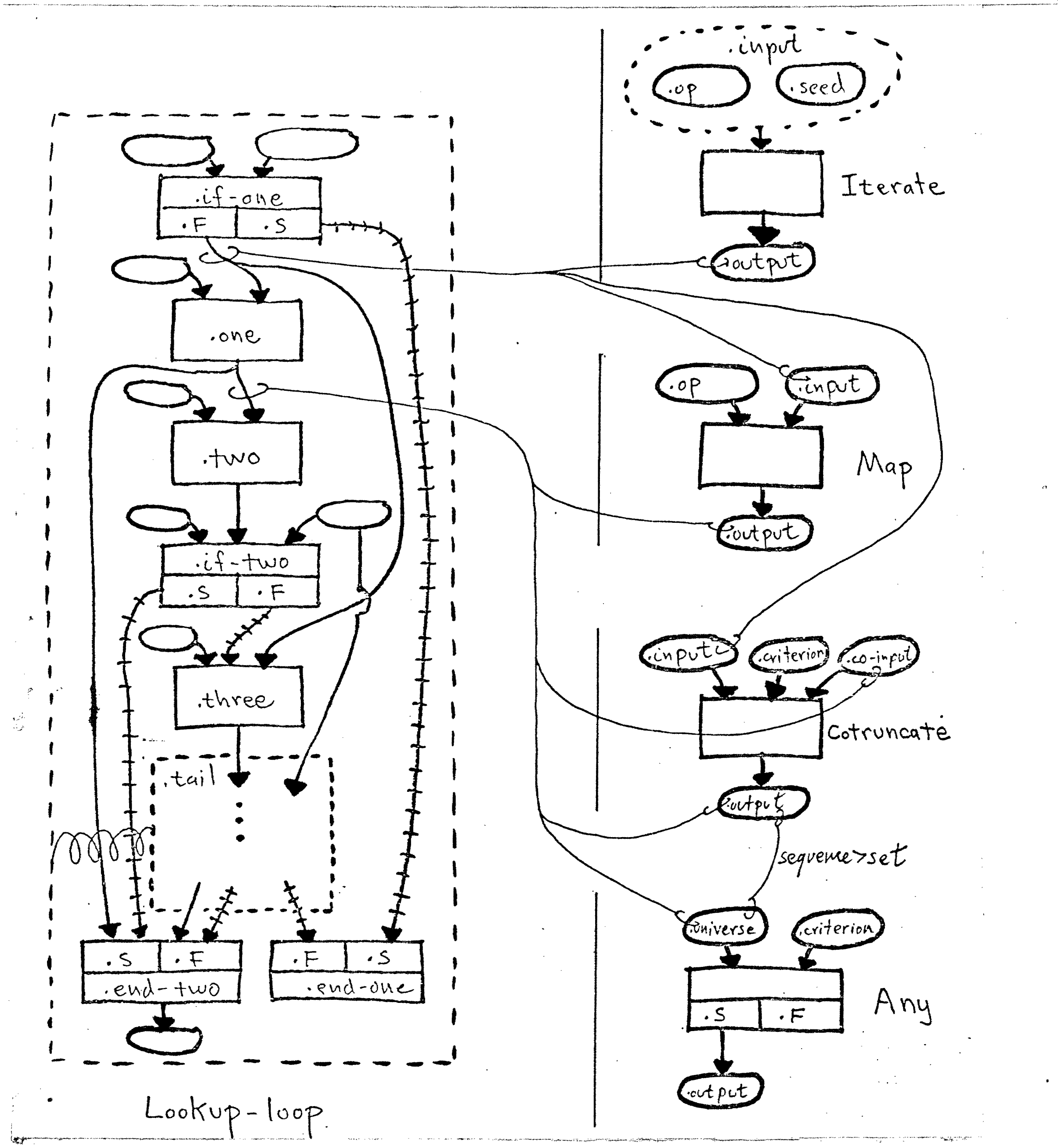


Figure 5-8. Temporal Abstractions of LOOKUP Loop.

The relationship between the output sequence of Cotruncate and Any.Universe is represented by an overlay, **Sequence>set**, which expresses in general how to view a sequence as a set. Such overlays between abstract descriptions are typical as analysis progresses beyond the surface plan.

In summary, Fig. 5-9 shows an overview of the plans and overlays used in the loop analysis thus far. The names of the plans are arranged in a hierarchy which reflects the order in which they must be recognized.[1] Each plan depends on the recognition of the plans below it as indicated by the vertical lines in the figure. Plans at the same level in the hierarchy may be recognized in any order. Overlays from the library used in this analysis are drawn as vertical lines with arrow heads to suggest that once the lower plan is recognized, the library is searched to suggest a more abstract description. The other lines represent pattern matching that is done specifically for this example. Notice that the analysis of a program is not strictly hierarchical. Distinct nodes at one level may share parts of the same plan at a level below. For example, the recognition of both Iteration and Iterative-application share the Iterative-steady-state plan. Conversely, the fact that a given plan or role has been used in one overlay, does not make it ineligible for use in others.

5.5 Bottom-up Recognition

It is natural to divide the analysis of LOOKUP roughly into three layers, as shown in Fig. 5-10. The bottom layer is loop analysis, as described in the preceding section. The middle and top layers are distinguished mostly by the complexity of the data structures involved. The plans in the middle layer involve only basic data structures such lists, sequences and sets. The effect of temporal abstraction, which is the final step of loop analysis, is to re-describe looping computations in terms of these basic data structures. The top layer of analysis in this example involves the relatively complex and specialized hash table data structure.

My intuition is that these general layers of abstraction are not specific to this example, though in larger programs there would be more upper layers. This means that the plan library itself can be roughly divided into layers. Most of the plans in the current library are in the middle layer involving lists, sequences, sets, and directed graphs. Presently the only more complicated data plans have to do with hash tables.

The three layers of description in this example also suggest a three phase strategy for automating analysis. The first phase is the specialized algorithm for loop analysis described in the preceding section. The second phase can be thought of as bottom-up pattern recognition, in which the standard plans involving basic data structures familiar to every experienced programmer are recognized. The third phase of analysis depends on being given some high level description of what the whole program is trying to do, so that top-down analysis by synthesis can be used. An alternative scenario, in which no top level description is given, is not considered here.[2] These three phases agree with my own introspection and

---

1. Some of these steps have been skipped over in this initial exposition, but are included here for future reference.
2. Such a scenario would presumably involve a much stronger control structure for hypothesis formation and testing.

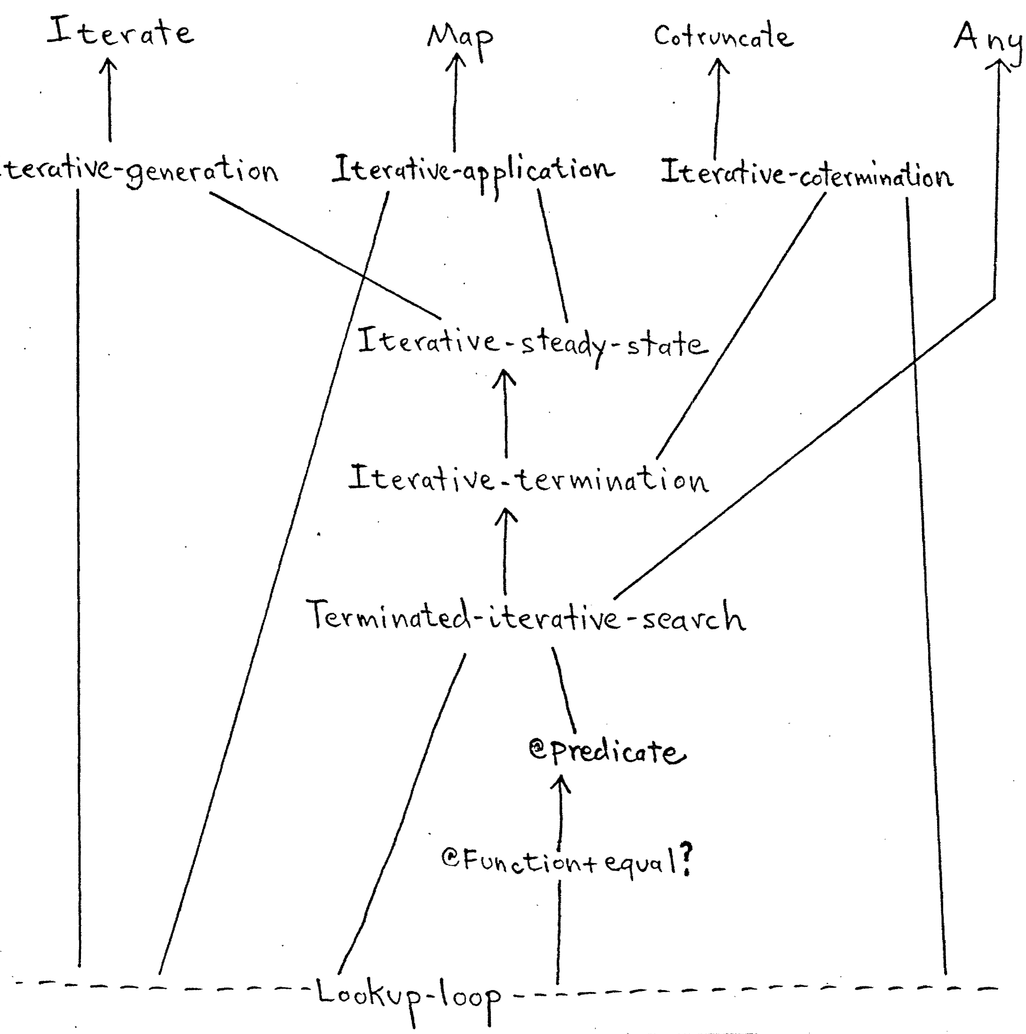


Figure 5-9. Overview of Analysis of LOOKUP Loop.

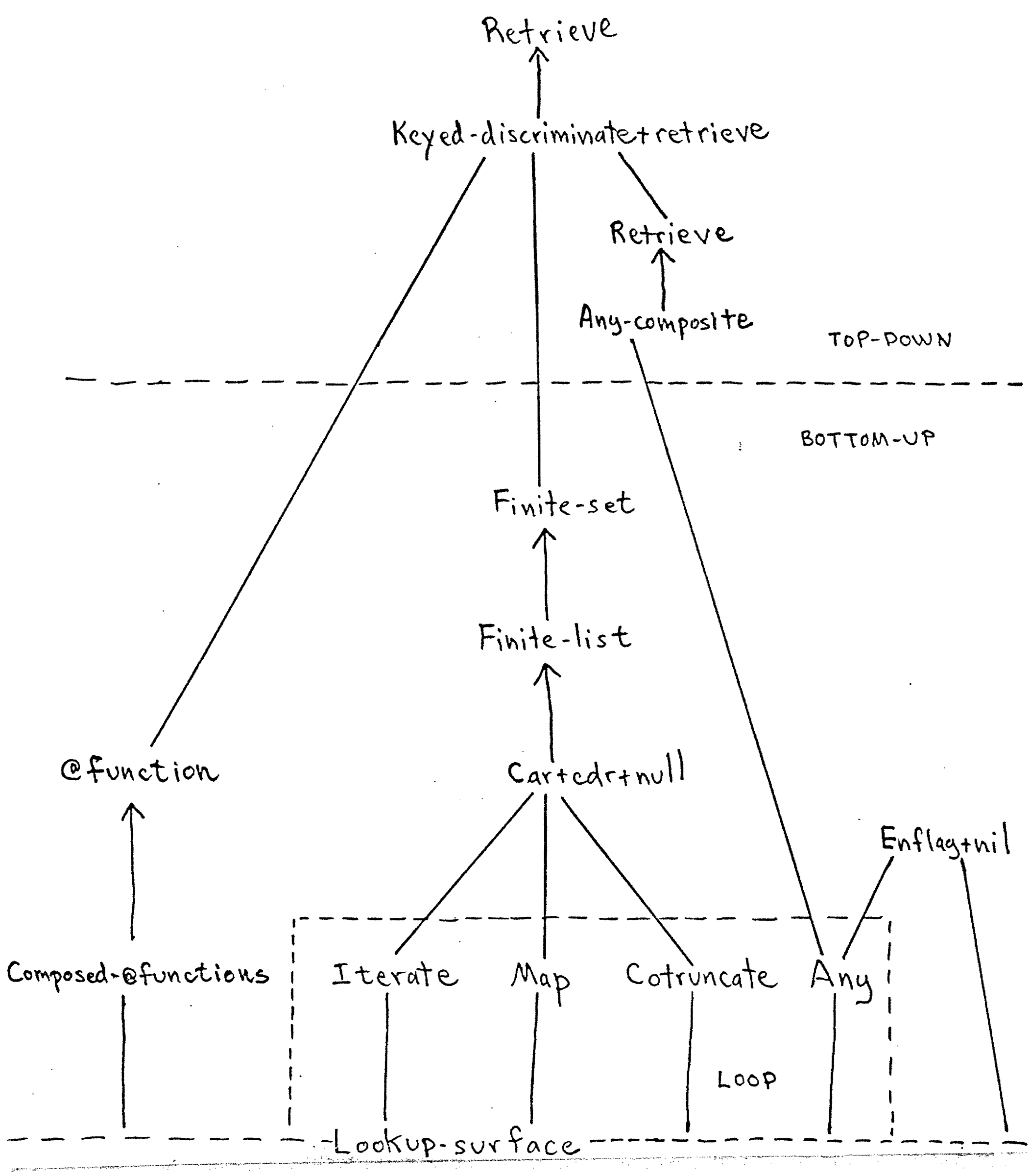


Figure 5-10. Layers in Analysis of LOOKUP.

experience in analyzing previously unseen programs. It would be interesting to conduct some experiments to verify the psychological validity of this model.

The rest of this section describes the particular plans in the middle layer of the analysis of LOOKUP, which are recognized bottom-up. The next section describes the plans in the top layer, which are recognized top-down, and how the two layers are connected.

As we can see in Fig. 5-10, there are several plans in the middle layer which may be recognized in any order. We begin with the plan **Car+cdr+null**, shown in Fig. 5-11, which has three steps: One (an instance Iterate), Two (an instance of Map) and Three (an instance of Cotruncate). The data flow between the roles in this plan is the same as between the overlays of Iterate, Map and Cotruncate on Lookup-loop described in the preceding section and shown in Fig. 5-8. This plan in general is called **Truncated-list-generation.** Car+cdr+null is a specialization in which the generating function is Cdr, the function being applied by Map is Car, and the criterion of Cotruncate is Null.[1]

Returning to LOOKUP, we have now come as far as recognizing that the initial input to the loop (the initial value of BKT) is a Lisp list. The temporal abstraction of the second exit from the loop as Any goes one step further and views this list as the implementation of a set. From the analysis of LOOKUP alone, it is not clear whether or not this list may contain duplicates. In the plan library, the implementation of sets as irredundant lists is represented as a specialization of the overlay used here.

There are two more small points to be covered. The first two steps of Lookup-surface (please refer back to Fig. 5-1) need to be analyzed as the application of a functional composition, Composed-@functions, described in Chapter Four. This is a common cliche which is needed here to put the surface plan in a form which will connect with the top-down recognition phase of the next section.

Another feature of the surface plan of LOOKUP to be recognized bottom-up is that the final output object (Lookup-surface.End.Output), which in the code is the value returned by the LAMBDA expression, can be viewed as a flag. Flags are a minor programming technique (formalized in the appendix). The basic idea is that the result of a test (in this case Any) is encoded in a data object so that the information discovered by the test can be recovered after a join. (Control is joined here because Lisp does not allow multiple return points.) The information encoded in the flag is recovered later by testing the object with a given predicate (Null in this case).

5.6 Top-down Recognition

The main point of this section to illustrate how a moderately complex data structure, such as a hash table, is decomposed in terms of the plan library. This section also introduces an important heuristic principle which is applicable to both analysis and synthesis by inspection.

The comment at the beginning of the code for the symbol table program in Table 5-A reads: "a set of entries is implemented as a hash table on keys". In the top-down part of this analysis scenario, we make use of this comment to retrieve the top few plans in the analysis of LOOKUP from the library.

1. We always try to recognize the most specialized version of a plan where possible.

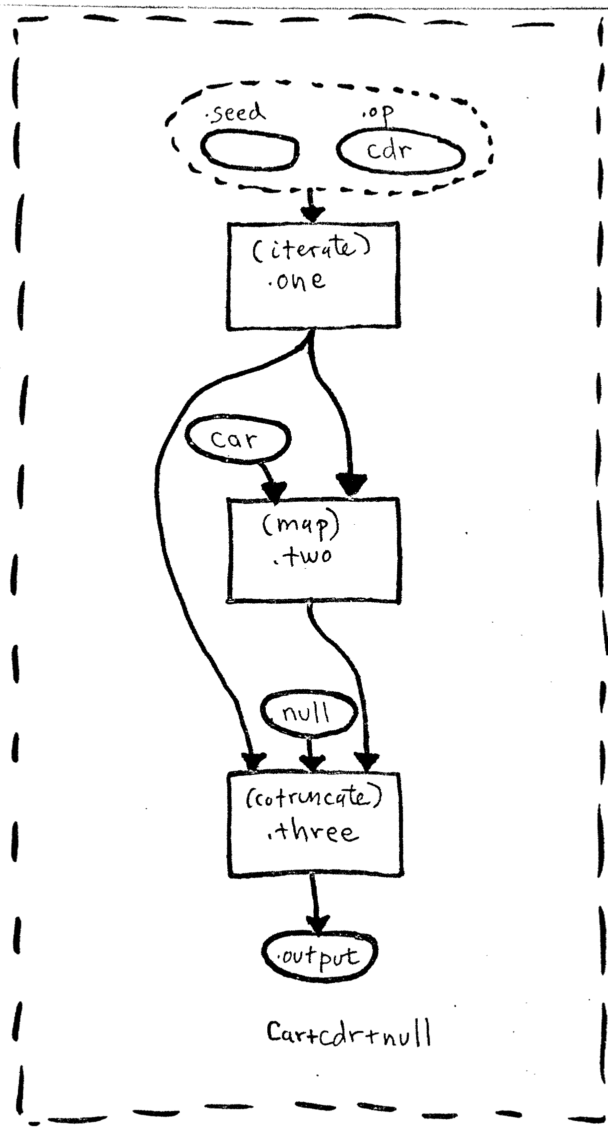


Figure 5-11. Plan for Generating a Lisp List.

At the highest level of abstraction, we are dealing with the implementation of a set. This set is implemented as a hash table on keys. In the analysis presented here, this implementation is decomposed into three basic ideas: discrimination, hashing and keys.

According to the plan library, a *discrimination* function maps some domain (in this example, "entries") onto a set of sets, called buckets. Such a function can be viewed as implementing a set wherein a given object is a member if and only if it is a member of the bucket obtained by applying the discrimination function to that object. Operations on a set implemented this way reduce to operations on a single bucket, which is often more efficient, especially in the case of operations which involve search. This idea is also part of many other data structures, such as discrimination nets.

The basic idea of *hashing* is to implement a discrimination function as the composition of two functions. The first function, called the hash function, maps the domain of the discrimination onto the set of valid indices for a sequence. The second function is a sequence, called the table, whose indices are the range of the hash function. The utility of this decomposition is that modifications to the discrimination function may be achieved by modifying only the table.

Discrimination on *keys* is also an implementation idea involving functional decomposition. In a keyed discrimination, each member of the implemented set has an associated key. In the symbol table example, the function from entries to keys is Car. The discrimination function in this implementation is the composition of two functions: the first function is the key function; the second function maps from the set of keys to the buckets. The utility of this decomposition is that for certain operations, such as associative retrieval, we are given only the key of an entry, rather than the entry itself.

To summarize, all three of these ideas are combined in the symbol table example as follows. At the top level we have a set implemented as a keyed discrimination. The key function is Car, and the function from keys to buckets is implemented as a hash table. The hash function of the hash table is Hash1 (HASH); the table is an abstraction of Vector1 (TBL), in which it is viewed as a a sequence of sets, each of which is implemented as Lisp lists.

Being able to formally analyze a data structure design in this way is a new and important result. This analysis gives a deep insight into the logical structure of this implementation and captures what it has in common with other implementations. It also decomposes the verification of the design, since each component can be separately verified. This aspect of the plan calculus is a contribution towards current efforts in computer science to develop an "algebra" of practical programming constructs. Others working in this effort have concentrated on the composition of procedural constructs [4], similar to the ideas described in the loop analysis section, or have worked only with simpler data structures [50].

The Maximal Sharing Heuristic

There are several different plausible accounts of how the analysis described above could be derived automatically, given the code and comments in Table 5-A, the bottom-up analysis described in the preceding section, and the current plan library. All of these accounts involve using what I call the *maximal sharing heuristic.* The origin of this heuristic is in program synthesis, but it also turns out to provide an elegant solution to the problem in program analysis of connecting bottom-up recognition with top-down analysis by synthesis.

In synthesis, the maximal sharing heuristic is applied at each implementation step. The basic idea of the heuristic is, rather than always adding new structure for an implementation, to reuse as many parts as possible of other plans in the current design which satisfy the constraints of the current implementation plan. The effect of this heuristic is to cause there to be a (locally) maximal amount of sharing in the analysis hierarchy. The motivations for this heuristic, and its application in synthesis are elaborated in Chapter Six.

The way to apply this heuristic in analysis is to view the parts of the bottom-up analysis as parts of the current design which are available for reuse. Whenever a part of the bottom-up analysis gets used in a top-down synthesis step, a connection has been achieved between the two phases of analysis. This holds out the promise that a module written for automated synthesis which obeys this heuristic may be used without change in automated analysis.

Another nice feature of this approach is that it suggests two fairly intuitive notions of partial analysis. One situation is when you can't find parts of the program you expect. This corresponds to when parts of the top-down synthesis never getting connected with the bottom-up analysis. In an interactive system, this could signal a potential bug or at least a request for further explanation from the user. The complementary situation is when parts of the bottom-up analysis are never used by the top-down phase. The most natural interpretation of this situation is that the programmer is using plans which are not in the current library. An interesting topic for future research is the possibility of isolating and generalizing these novel parts of a program so that new plans can automatically be added to the library.

Returning now to LOOKUP, let us follow one account of how the final steps of analysis might proceed. It may help to refer to Fig. 5-10 to follow this explanation.

The first step in the top-down analysis is to conclude that the set operation implemented by LOOKUP is associative retrieval. This could be deduced from the name of the procedure, or by looking at the types of its inputs and outputs and the fact that it has two cases.

The library overlay for implementing associative retrieval from a keyed discrimination is shown in Fig. 5-12. The input-output specification for associative retrieval on the right hand side is called **Retrieve.** It is a test with three inputs, a set (Universe), the key function (Key), and an input key (Input), and one output. If there exists a member of the set with the given key, then the test succeeds and returns such a member; otherwise it fails. On the left hand side of the overlay we have the typical two step plan for implementing a set operation on a discrimination: apply the discrimination function to fetch the appropriate bucket, and then perform the same operation on the bucket. This general implementation works for adding and removing a member, and certain kinds of retrieval. It does not work for other operations such as union or intersection.

The first step (Discriminate) of the plan on the left of Fig. 5-12 is thus constrained to be an instance of @Function, in which the function being applied is the discrimination function from keys to buckets. The maximal sharing heuristic suggests using the @Function recognized bottom-up (see Fig. 5-10) in this role. Recall that this @Function is itself implemented as the composition of two instances of @Function,

```
(ARRAYFETCH TBL (HASH KEY)),
```

from which we can conclude that the hashing function is Hash1 and the table is Vector1.

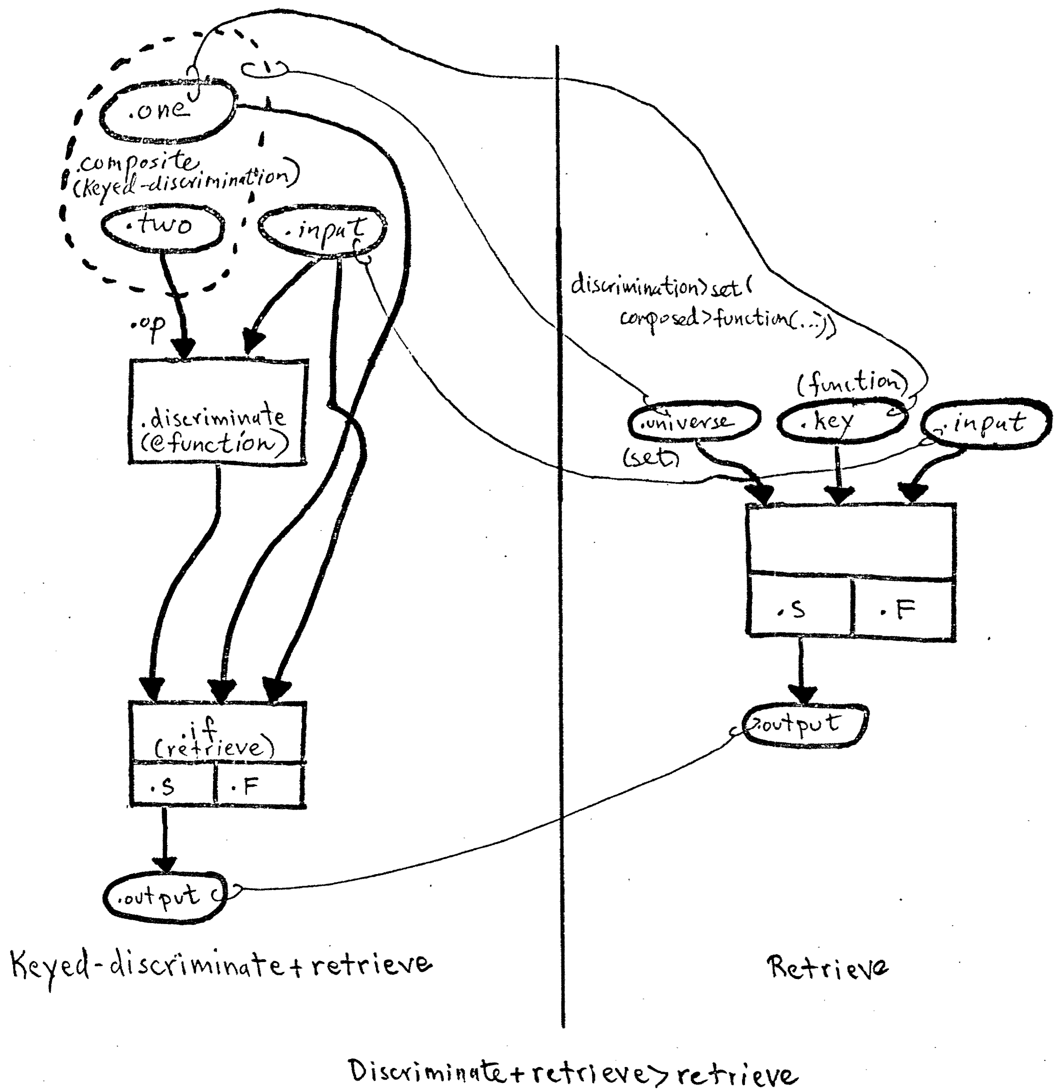


Figure 5-12. Associative Retrieval from a Keyed Discrimination.

The second step (If) of the plan on the left of Fig. 5-12 is constrained to be an instance of Retrieve, applied to the bucket fetched in step One. According to the second comment at the front of the code (see Table 5-A), "the buckets are implemented as lists". The only implementation in the library for Retrieve on sets other than those implemented as discriminations is as Any (an input-output specification introduced earlier in this chapter) in which the criterion is a composite predicate. The form of this predicate is to test whether the key of a given object is equal to some constant. If the key function is Car, this comes down to Lisp code like the following test in LOOKUP.

```
(COND ((EQ (CAR ...) KEY)
       ...))
```

The sharing heuristic suggests recognizing the bottom-up Any specification as implementing the bucket Retrieve in this way. In order for this to be the case, the key function of the hashing implementation must be Car.

This completes the analysis of LOOKUP. Let me emphasize that the last few paragraphs are only one of many possible accounts of how the top-down recognition could be accomplished. There are many other strong clues in this program, particularly in the types of objects. For example, the only candidate for the table part of the hash table (by virtue of being a vector) is Vector1; the only candidate for the hash function (by virtue of being a numerical function) is Hash1.

CHAPTER SIX

SYNTHESIS BY INSPECTION

6.1 Introduction

A library of plans, such as presented in Chapter Three, opens up many new possibilities for what an interactive program development system can do to help a user synthesize programs. This chapter is an exploration into some of these new possibilities. This chapter also shows the use of the plan calculus as a design language, and picks up where Chapters Two and Five leave off in showing how the plans in the current library are used together in a complete example.

In broad terms, the plan library represents a significant body of knowledge about programming which is shared between the user and the system, which has never been the case before. The most advanced current program development systems (e.g. [71,14]) have some built-in knowledge of programming language syntax and type restrictions, but none include the range or kind of knowledge represented in the plan library.

This chapter presents a simple scenario of interactive program synthesis in which the working medium is the plan calculus rather than Lisp code. Code is generated only as a final translation of the synthesized surface plan. The order of development in this scenario is primarily top-down. The user progressively refines an initial abstract specification by application of overlays from the plan library. This scenario is thus restricted to programs which can be completely analyzed using plans in the library alone. This scenario also portrays an expert user who is familiar with the plan library.

The fundamental interaction between the system and the user in this scenario is for the system to propose a *menu* of overlays from the library which are applicable to the current design plan, and for the user to choose between them. In this way, the user guides the synthesis in top-down fashion. The user also intervenes at certain crucial points in the development to introduce new plans from the library, and to suggest reanalysis of the current design which leads to a more efficient implementation. In addition to retrieving overlays from the library, the system also needs to be able to spontaneously propagate some information and construct specializations of library plans appropriate to the current design. This implies a deductive component in the system, whose operation will not be discussed here, since it is part of related research reported elsewhere [64].

Deductive capabilities are also required to apply the maximal sharing heuristic. The basic idea of this heuristic, as described in Chapter Five, is to build plans which share as much structure as possible.[1] The motivation for this heuristic is that it often leads to more efficient programs. It is applied in synthesis each time an overlay is used to further implement some part of the current design. To apply the heuristic,

1. Sacerdoti, in his work on general problem solving [60] uses a similar heuristic of the form "use existing objects whenever possible".

the system needs to know whether a subset of the roles on the left hand side of an overlay can be identified with roles of other existing plans in the current design, while being consistent with both the constraints of the plan on the left hand side of the overlay and the existing constraints on the other roles.

With the maximal sharing heuristic in operation, synthesis using overlays becomes a mixture of progressive refinement and constraint. In the refinement steps, an overlay is used to expand the current design in a tree-like fashion, by adding more detail at one of the terminal nodes. Alternatively, whenever sharing is established in the application of an overlay, the effect to is add further constraints to the current design.

The synthesis scenario in this section is divided into three distinct phases: data structure design, procedure implementation, and code generation. In the first phase, the user lays out the implementation of the hash table data structure using overlays between data plans. The second phase, procedure implementation, involves refining the input-output specifications for associative retrieval, addition, and deletion on the hash table down to the level of Lisp surface plans. The final phase is the generation of Lisp code from surface plans.

The major purpose of this scenario is to demonstrate what it could be like to develop programs interactively with a system that had significant programming knowledge in the form of a plan library. The particular order of development is not in any way canonical. Any realistic such system will have to be based on a mixed initiative model which allows the user to tailor the order of development to the particular programming task at hand. As in the scenario of Chapter Two, lines typed by the system are shown in upper case; lines typed by the user are shown in lower case.

6.2 Data Structure Design

In this section, the user designs the main data structure of the symbol table program, starting with a description of it as a set of entries, and culminating with its implementation as a keyed discrimination in which the function from keys to buckets is implemented by hashing.

```
> let an "entry" be a data structure.
> "symbol" (an atom) is part of an entry.
> "info" is part of an entry.
```

The user begins by defining a new data plan which is particular to the programming task at hand. This definition becomes a part of the permanent documentation of the program. An "entry" is defined as a data structure with two fields, named Symbol and Info. The Symbol field contains a Lisp atom; nothing is said about the Info field.

```
> let a "symbol table" be a finite set.
> for.all "x" if x belongs to a symbol table, then x is an entry.
```

The user now defines a symbol table as finite set of entries. The rest of this section is concerned with implementing this set.

```
> ↵
APPLICABLE IMPLEMENTATIONS FOR: SYMBOL-TABLE (A FINITE-SET)
      1. LIST
      2. SEQUENCE
      3. LABELLED-THREAD
      4. DISCRIMINATION FUNCTION
? 4
```

The user transfers initiative temporarily to the system by typing an empty command line. The basic action of the system in this situation is retrieve from the library all overlays which are applicable to the current design. In general, this includes all overlays whose right hand sides (or specializations or extensions of them) appear in the current design. Furthermore, the system in general first proposes overlays for a whole plan before trying to find overlays for any of its roles. In this case, the system retrieves four overlays: the first three are for implementing sets as lists, sequences, and labelled threads, respectively; the fourth is for implementing a finite set as a discrimination. The user chooses the fourth overlay as the one to apply, and the system responds by instantiating the left hand side of the overlay, propagating the appropriate information through the correspondences.

```
LET "DISCRIMINATION1" BE THE IMPLEMENTATION OF SYMBOL-TABLE AS A
DISCRIMINATION FUNCTION.
```

Thus Discrimination1 is a discrimination function on entries, which is the implementation of Symbol-table. Fig. 6-1 shows the complete implementation of Symbol-table, as it will be developed in this section. Implementation relationships, such as between Discrimination1 and Symbol-table, are shown in this figure as vertical arrows pointing up from the instance of the left hand side to the instance of the right hand side. (This is the same convention as was used in Chapter Five.) In this figure, and the other similar one is this chapter, vertical lines without arrow heads denote part-whole relationships, with the role names indicated.

```
> ↵
APPLICABLE IMPLEMENTATIONS FOR: DISCRIMINATION1 (A DISCRIMINATION)
      1. COMPOSED-FUNCTIONS
      2. UNARY-LAMBDA
? 2
```

The scenario continues with the system proposing applicable overlays for Discrimination1. In this case, the only applicable overlays are for functions in general, of which there are two in the current library: a function can either be implemented as the composition of two other functions, or it can be implemented directly in Lisp as a lambda expression. The user chooses the first option.

```
SPECIALIZATIONS OF: COMPOSED-FUNCTIONS
      1. HASHING
      2. KEYED-DISCRIMINATION
? 2

LET "KEYED-DISCRIMINATION1" BE THE IMPLEMENTATION OF DISCRIMINATION1 AS A
KEYED-DISCRIMINATION.
```

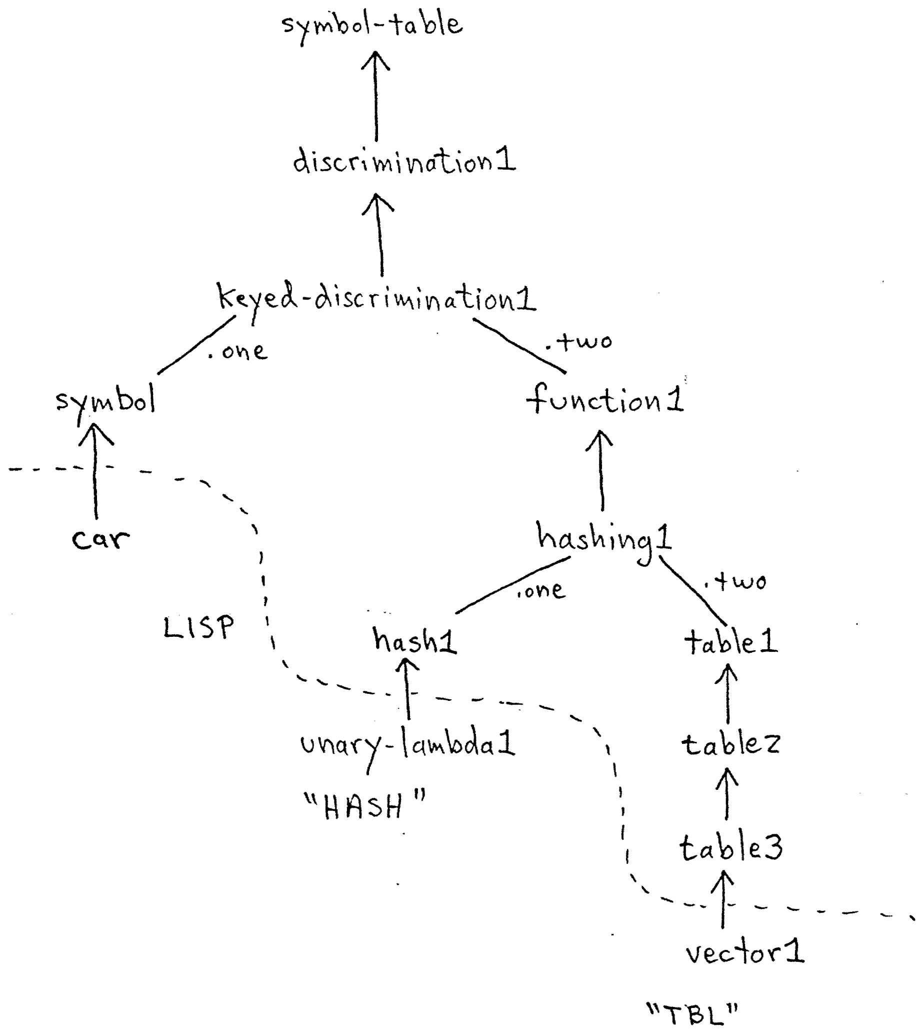


Figure 6-1. Implementation of the Symbol Table.

Before applying an overlay, the system checks in the library for any specializations or extensions of the plan on the left hand side. If any exist, the user is given a chance to choose one. In this case, the system finds two specializations of Composed-functions, namely Hashing and Keyed-discrimination. The user chooses the second, causing the system to instantiate Keyed-discrimination1 as the implementation of Discrimination1. (The user could have responded "no", causing the system to instantiate an instance of Composed-functions with no further constraints.)

```
> )
ELIGIBLE SHARING FOR: KEYED-DISCRIMINATION1.ONE ("THE KEY FUNCTION")
      1. SYMBOL
      2. INFO
? 1

LET SYMBOL BE THE KEY FUNCTION OF KEYED-DISCRIMINATION1.
LET "FUNCTION1" THE BUCKET FUNCTION OF KEYED-DISCRIMINATION1.
```

The instantiation of Keyed-discrimination1 gives us our first opportunity to see the maximal sharing heuristic in action. The system above has searched for existing objects in the current design which could fill the roles of Keyed-discrimination1 and satisfy the constraints of the Keyed-discrimination plan. The first filter on this search can be the object *types* — roles One and Two of a keyed discrimination must be functions. There are three functions in the current design: Symbol,[1] Info and Discrimination1. Discrimination1 can immediately be eliminated from consideration because it is above Keyed-discrimination1 in the refinement tree, so that sharing with it would lead to a meaningless circularity. Symbol and Info can be rejected for role Two of Keyed-discrimination1, since the range of this function is constrained to be finite sets. This leaves the possibility of Symbol or Info filling role One of Keyed-discrimination, which the system proposes as shown above. The user chooses a keyed discrimination on the symbol field of entries. The system completes this frame of the interaction by instantiating Function1, a function from Lisp atoms to finite sets, to fill role Two. (Again, the user could have responded "no" to the question above, in which case a new object would be instantiated for role One as well as for role Two.)

```
> )
APPLICABLE IMPLEMENTATIONS FOR: FUNCTION1 (A FUNCTION)
      1. COMPOSED-FUNCTIONS
      2. UNARY-LAMBDA
? 1

SPECIALIZATIONS OF: COMPOSED-FUNCTIONS
      1. HASHING
      2. KEYED-DISCRIMINATION
? 1

LET "HASHING1" BE THE IMPLEMENTATION OF FUNCTION1 AS A HASHING.
```

In this next frame, Function1 is implemented as a hash table, Hashing1.

1. Role names are formally functions.

```
> )
LET "HASH1" BE THE HASH FUNCTION OF HASHING1.
LET "TABLE1" BE THE TABLE OF HASHING1.
```

Since there are no existing objects which can fill the roles of Hashing1, the system instantiates Hash1 and Table1.

```
> implement the buckets of table1.
```

After letting the system carry the initiative for a few steps, the user intervenes here with a command to retrieve overlays from the library for implementing the buckets of the discrimination (the range elements of Table1).

```
APPLICABLE IMPLEMENTATIONS FOR: BUCKETS OF TABLE1 (A FINITE-SET)
      1. LIST
      2. SEQUENCE
      3. LABELLED-THREAD
      4. DISCRIMINATION
? 1

SPECIALIZATIONS OF: LIST
      1. IRREDUNDANT-LIST
? 1

LET "TABLE2" BE THE IMPLEMENTATION OF TABLE1 SUCH THAT
FOR ALL "I" AND "X", IF TERM I OF TABLE2 EQUALS X, THEN X IS THE
IMPLEMENTATION OF TERM I OF TABLE1 AS AN IRREDUNDANT-LIST.
```

The user above chooses to implement the buckets as lists, and in particular lists without duplicates. The system responds by instantiating Table2 which is related to Table1 an overlay in which each term of Table2 (an irredundant list) is viewed as the corresponding term of Table1 (a finite set).

This completes the part of the data structure design that is programming language independent (see Fig. 6-1).

Lisp-Specific Data Structure Implementation

The remaining few steps in this section are involving with bringing the data structure design down to the level of Lisp data objects.

```
> implement an entry as a dotted pair, such that
  entry.symbol=dotted-pair.car and entry.info=dotted-pair.cdr
```

Here the user specifies the implementation of entries as dotted pairs such that the Symbol function is implemented by Car and the Info function by Cdr.

```
> )
APPLICABLE IMPLEMENTATIONS FOR: HASH1 (A HASHING)
      1. COMPOSED-FUNCTIONS
      2. UNARY-LAMBDA
? 2
```

Hash1 is to be implemented as a Lisp procedure, which will be coded later.

```
> )
APPLICABLE IMPLEMENTATIONS FOR: BUCKETS OF TABLE2 (AN IRREDUNDANT-LIST)
      1. UPPER-SEGMENT
      2. DOTTED-PAIR
      3. LIST-GENERATION
? 2

LET "TABLE3" BE THE IMPLEMENTATION OF TABLE2 SUCH THAT
FOR ALL "I" AND "X", IF TERM I OF TABLE3 EQUALS X, THEN X IS THE
IMPLEMENTATION OF TERM I OF TABLE2 AS A DOTTED-PAIR.
```

The range elements (buckets) of Table2 are not yet implemented as Lisp data objects. The system suggests three overlays for implementing lists. The user chooses the direct route of implementing lists as dotted pairs.

```
> )
APPLICABLE IMPLEMENTATIONS FOR: TABLE3 (A SEQUENCE)
      1. SEGMENT
      2. VECTOR
      3. LIST

? 2

LET "VECTOR1" BE THE IMPLEMENTATION OF TABLE3 AS A VECTOR.
```

Finally we implement Table3, a sequence of dotted pairs, as a Lisp vector.

6.3 Procedure Synthesis

The user now moves on to the implementation of some procedures which access the symbol table data structure. The first procedure is to retrieve the entry associated with a given symbol. Fig. 6-2 gives an overview of this implementation. Down the left side of this figure is the data structure implementation developed in the preceding section. As in Fig. 6-1, arrows in this figure denote overlays and roles names are labelled. In this figure, however, many roles are left out in order to make it more readable. The names in parentheses are the types of the roles.

```
> let "symbol table retrieve" be a specialization of retrieve,
  such that the universe is a symbol table, and the key function is symbol.
```

The starting point for the program development is to specialize the library input-output specification Retrieve by constraining the Universe to be a symbol table and the key function to be the Symbol function defined earlier in the scenario. In conventional terms, this would be called the "specifications" of the procedure. It is important to note, however, that in the plan calculus the usual distinction between specifications and implementations as separate formalisms does not exist. What we have in general is plans at various levels of abstraction. The topmost plan often amounts to what would normally called a specification, and the bottommost (surface) plan is certainly what would be called an implementation. All of these descriptions are in the same language, and there are implementation relationships between the intermediate plans also. Furthermore, in this framework there is no reason to restrict a user's starting plan to being an input-output specification. The most natural starting description may sometimes be a multi-step plan.

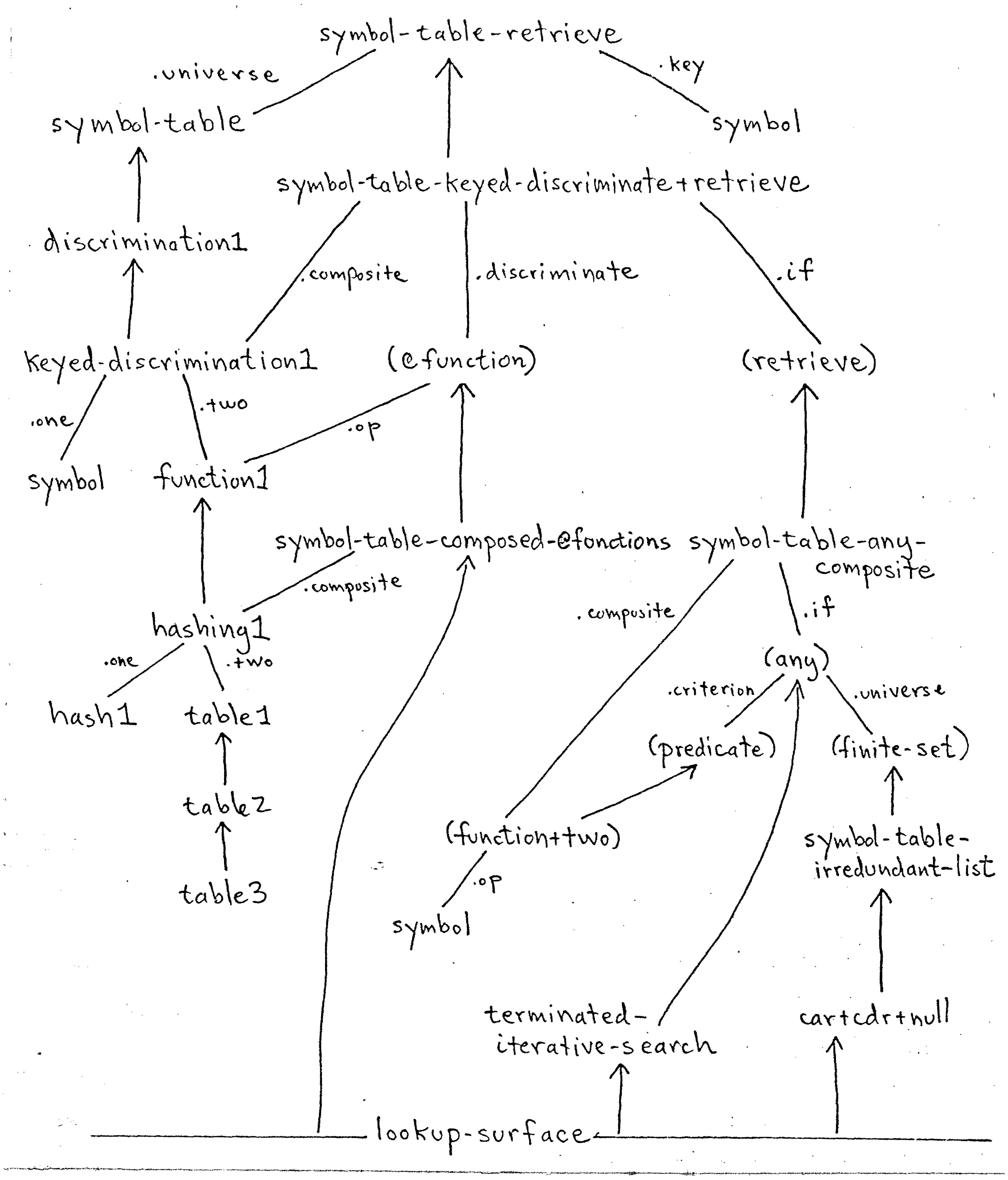


Figure 6-2. Implementation of Symbol Table Retrieval.

```
> ↵
APPLICABLE IMPLEMENTATIONS FOR: SYMBOL-TABLE-RETRIEVE (A RETRIEVE)
      1. ANY-COMPOSITE
      2. DISCRIMINATE+RETRIEVE
? 2
```

In the interaction above, the system has searched the plan library for ways of implementing Symbol-table-retrieve (i.e. for overlays with Retrieve as their right hand side). In the current library, there are two: the default implementation as Any, and the implementation in which the universe is implemented as a keyed discrimination (see Fig. 6-3). These are presented as options to the user, who chooses the second.

The system at this point could have been more clever and concluded that the second choice was indicated, since Symbol-table has already been implemented as Keyed-discrimination1. However, this degree of automation in general may be difficult, particularly in the presence of multiple views. In any case, once option two is chosen, either by the system or the user, the maximal sharing heuristic ensures that Keyed-discrimination1 does become part of the implementation of Symbol-table-retrieve.

```
LET "SYMBOL-TABLE-KEYED-DISCRIMINATE+RETRIEVE" BE THE IMPLEMENTATION
OF SYMBOL-TABLE-RETRIEVE AS KEYED-DISCRIMINATE+RETRIEVE.

ELIGIBLE SHARING FOR: SYMBOL-TABLE-KEYED-DISCRIMINATE+RETRIEVE.COMPOSITE
                                        (A KEYED-DISCRIMINATION)
      1. KEYED-DISCRIMINATION1
? ↵
```

The system has created a specialized version of the plan Keyed-discriminate+retrieve, (wherein the keyed discrimination is Keyed-discrimination1) which implements Symbol-table-retrieve (see Fig. 6-2).

```
> ↵
APPLICABLE IMPLEMENTATIONS FOR:
            SYMBOL-TABLE-KEYED-DISCRIMINATE+RETRIEVE.DISCRIMINATE (AN @FUNCTION)
      1. COMPOSED-@FUNCTIONS
? ↵

LET "SYMBOL-TABLE-COMPOSED-@FUNCTIONS" BE THE IMPLEMENTATION OF
SYMBOL-TABLE-KEYED-DISCRIMINATE+RETRIEVE.DISCRIMINATE AS COMPOSED-@FUNCTIONS.

ELIGIBLE SHARING FOR:
              SYMBOL-TABLE-COMPOSED-@FUNCTIONS.COMPOSITE (A COMPOSED-FUNCTIONS)
      1. HASHING1
? ↵
```

Since there are no overlays for implementing Symbol-table-keyed-discriminate+retrieve as a whole, the system proposes applicable overlays for the roles, beginning with the Discriminate role, which is constrained to be an instance of @Function. There is only one plan in the library for implementing @Function, i.e. as a composition of two other instances of @Function. Using the maximal sharing heuristic again, these become the application of the hash function, Hash1, followed by fetching from the hash table, Table1.

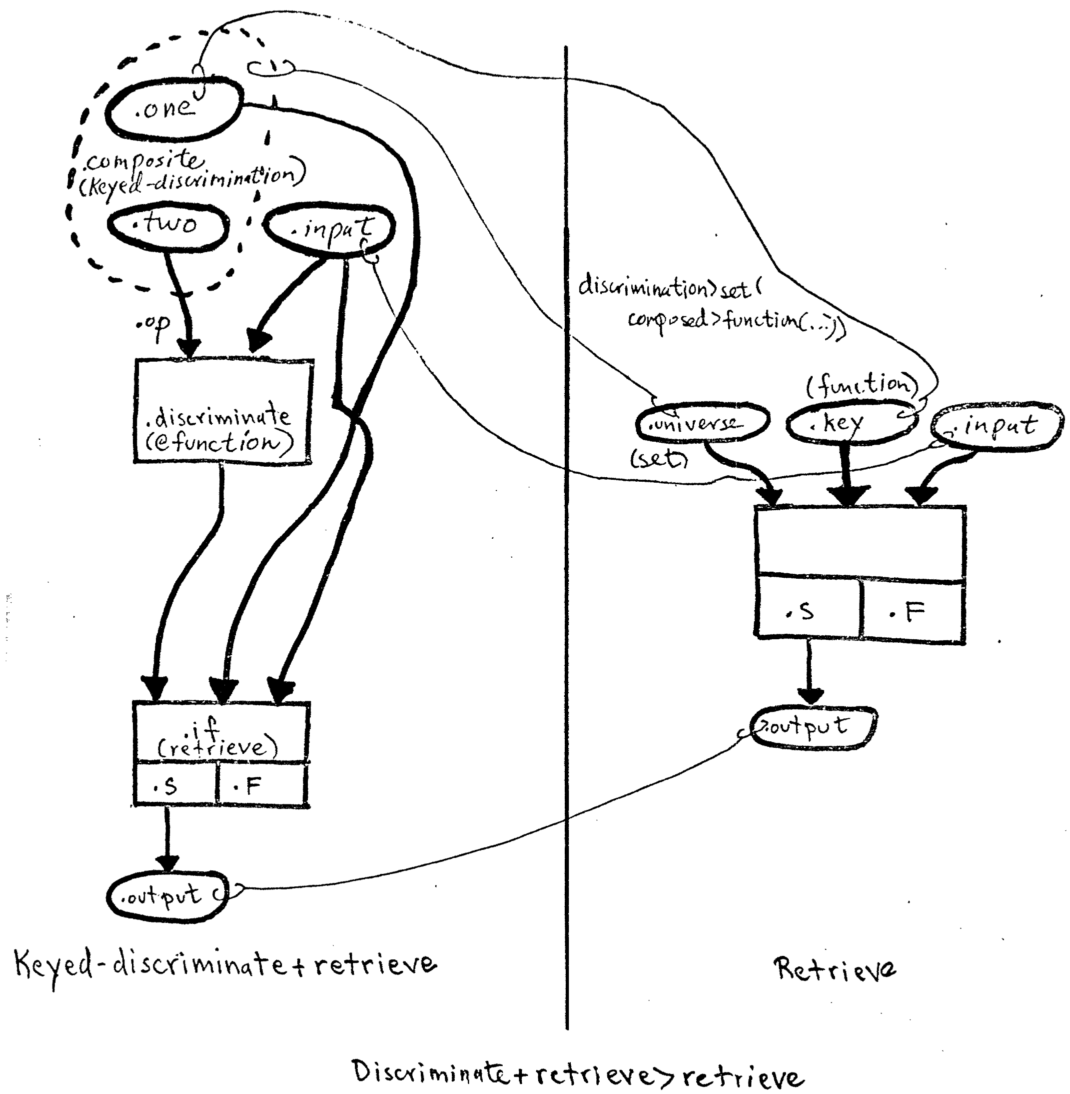


Figure 6-3. Associative Retrieval from a Keyed Discrimination.

```
> ↲
APPLICABLE IMPLEMENTATIONS FOR:
                SYMBOL-TABLE-KEYED-DISCRIMINATE+RETRIEVE.IF (A RETRIEVE)
      1. ANY-COMPOSITE
      2. DISCRIMINATE+RETRIEVE
? 1

LET "SYMBOL-TABLE-ANY-COMPOSITE" BE THE IMPLEMENTATION OF
SYMBOL-TABLE-KEYED-DISCRIMINATE+RETRIEVE.IF AS ANY-COMPOSITE.
```

Implementation of the other role (If) of Symbol-table-keyed-discriminate+retrieve is shown above. This role is an instance of Retrieve applied to the bucket obtained by Discriminate. As before, the system presents two options for implementing Retrieve. This time the user chooses the first option: retrieval from the bucket is implemented as Any in which the criterion is a composite of the key function (Symbol) and the Input to Retrieve. This is the default way of implementing Retrieve. For example, if the Universe set is implemented as a list, Retrieve will typically be implemented as a CAR-CDR search loop.

The overlay for this implementation is shown in Fig. 6-4. The plan on the left hand side is called **Any-composite.** This overlay shows how an instance of Retrieve can be implemented as an instance of Any in which the Criterion predicate has a definition of the following form.

$$P(x) \equiv F(x,K)$$

This way in general of constructing a predicate, P, for a given function, F, and a value, K, is formalized as the overlay **Function+value>predicate** (see appendix). In the implementation of Retrieve as Any, F corresponds to the key function of Retrieve and K is the input key.

Loop Synthesis

We now come to the point in the synthesis where loops are introduced into the design. Note that the maximal sharing heuristic also applies to loops, i.e. for efficiency, the loop implementations of different parts of a program should be combined into a single loop when possible. In order to achieve this part of the scenario below, an additional temporal synthesis module (the inverse of the temporal analysis module discussed in Chapter Five) is needed.[1]

```
> ↲
APPLICABLE IMPLEMENTATIONS FOR: SYMBOL-TABLE-ANY-COMPOSITE.IF (AN ANY)
      1. TERMINATED-ITERATIVE-SEARCH
? ↲
```

To begin, Any is implemented as Terminated-iterative-search (this overlay was already discussed in Chapter Five). The Universe of Any is then implemented as a loop augmentation which generates the inputs to the second (success) exit test of this search loop. This takes place in two steps under user guidance. In the first step, shown below, the set is implemented as a list (without duplicates).

1. Waters has written a module which does part of this work.

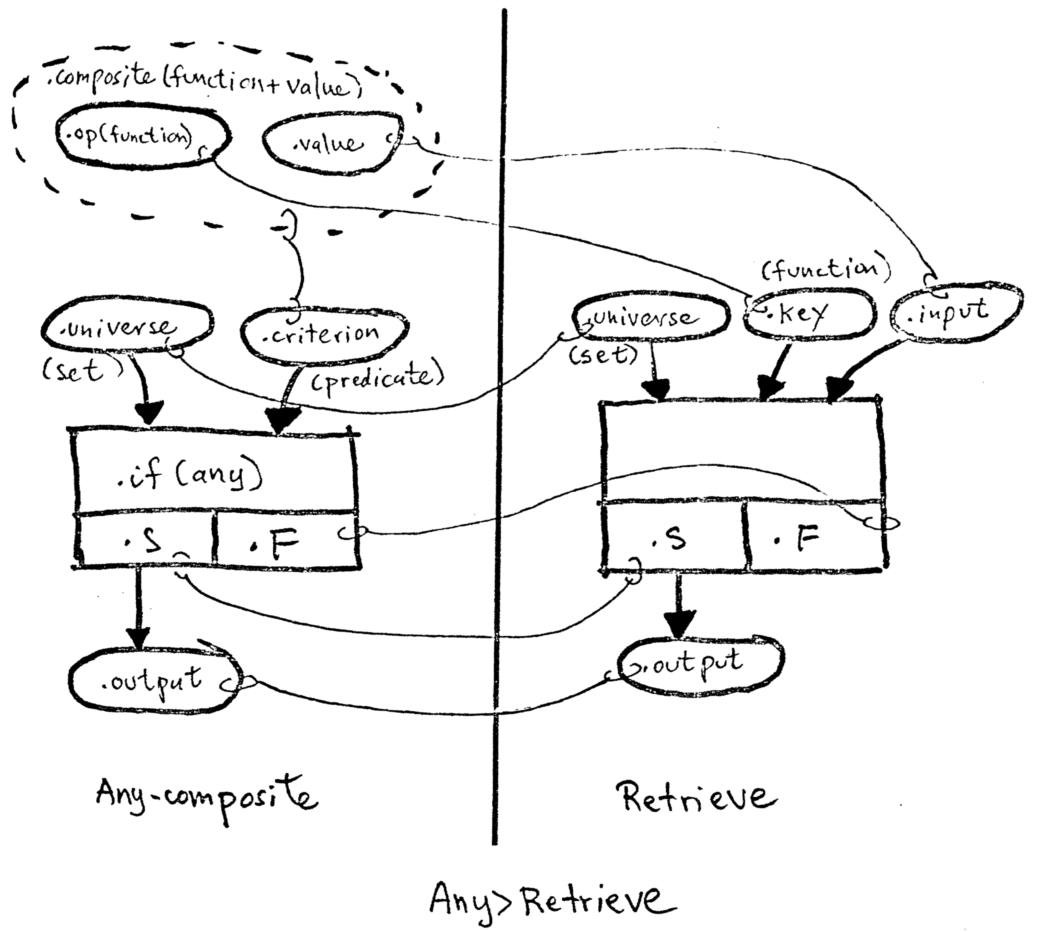


Figure 6-4. Default Implementation of Associative Retrieval.

```
> ↲
APPLICABLE IMPLEMENTATIONS FOR:
                SYMBOL-TABLE-ANY-COMPOSITE.IF.UNIVERSE (A FINITE-SET)
        1. LIST
        2. SEQUENCE
        3. LABELLED-THREAD
        4. DISCRIMINATION
? 1

SPECIALIZATIONS OF: LIST
        1. IRREDUNDANT-LIST
? ↲

LET "SYMBOL-TABLE-IRREDUNDANT-LIST" BE THE IMPLEMENTATION OF
SYMBOL-TABLE-ANY-COMPOSITE.IF.UNIVERSE AS AN IRREDUNDANT-LIST.
```

As earlier in this scenario, the system could also be more clever here and save the user some effort. In particular, the system should realize that the finite sets which fill the Symbol-table-any-composite.If.Universe role are the same as the buckets of Table1, which were implemented earlier as irredundant lists. Finally, the irredundant lists of entries are implemented as the standard Lisp CAR, CDR and NULL loop, as shown below.

```
> ↲
APPLICABLE IMPLEMENTATIONS FOR:
                SYMBOL-TABLE-IRREDUNDANT-LIST (AN IRREDUNDANT-LIST)
        1. UPPER-SEGMENT
        2. DOTTED-PAIR
        3. LIST-GENERATION
? 3

EXTENSIONS OF: LIST-GENERATION
        1. TRUNCATED-LIST-GENERATION
? ↲

SPECIALIZATIONS OF: TRUNCATED-LIST-GENERATION
        1. CAR+CDR+NULL
? ↲
```

Code Generation

The final phase of top-down synthesis is code generation. The user has guided the implementation of all parts of the program down to the level of input-output and test specifications involving the primitive functions and relations of Lisp. All that remains now to obtain a complete surface plan is to gather up all the control flow and data flow constraints between terminal nodes of Fig. 6-2 into a single non-hierarchical plan (and also to add additional, arbitrarily chosen, control flow arcs as necessary to totally order any otherwise unordered steps). The resulting surface plan is essentially the same as the surface plan obtained in the analysis of LOOKUP in Chapter Five. This surface plan is then turned over to a code generator, which implements the data and control flow using the available connective primitives in the programming language, as shown below.

```
> show code for symbol-table-retrieve

        (DEFINE SYMBOL-TABLE-RETRIEVE
           (LAMBDA (TABLE INPUT)
              (PROG (BUCKET OUTPUT)
                    (SETQ BUCKET (ARRAYFETCH TABLE (HASH INPUT)))
                 LP (COND ((NULL BUCKET)(RETURN NIL)))
                    (SETQ OUTPUT (CAR BUCKET))
                    (COND ((EQ (CAR OUTPUT) INPUT)
                           (RETURN OUTPUT)))
                    (SETQ BUCKET (CDR BUCKET))
                    (GO LP))))
```

A code generator for Lisp has been implemented by Waters [74]. Note that at the end of this entire process, we get essentially the same structure as in Chapter Five, namely code for the program (perhaps with some minor syntactic variations due to the stylistic biases of the code generator), together with a complete hierarchical decomposition of the design in terms of plans in the library.

Synthesis of Symbol Table Addition

This section shows the synthesis of a procedure to add entries to the symbol table. Two new points are introduced in this example. First, the plans in this example involve side effects. Second, the user intervenes at a key point in the development in order to suggest a reanalysis which leads the system to the desired program. An overview of the complete implementation structure is shown in Fig. 6-5.

```
> let "symbol table add" be a specialization of set add by side effect
  such that the old set is a symbol table, and the input does not belong to
  the old set.
```

The starting point for this synthesis is a specialization of the input-output specification **Set-add,** in which the old set is a symbol table. The first additional constraint above specializes Set-add to the side effect version, **#Set-add,**[1] in which the new set has the same identity as the old set (but different members). The role names Old and New in this case refer to the different states of the same set before and after the side effect operation, rather than to different sets.[2] The user has also specified as a precondition that the entry to be added is not already in the table. This is another standard specialization of Set-add, called **Set-add-one**, which has simpler implementations in which there is no need to check for duplicates.

1. The character "#" is intended to be read as "impure". Thus #Set-add is "impure set add" or "set add by side effect".
2. The formal representation of side effects will be specified in more detail in Chapters Eight.

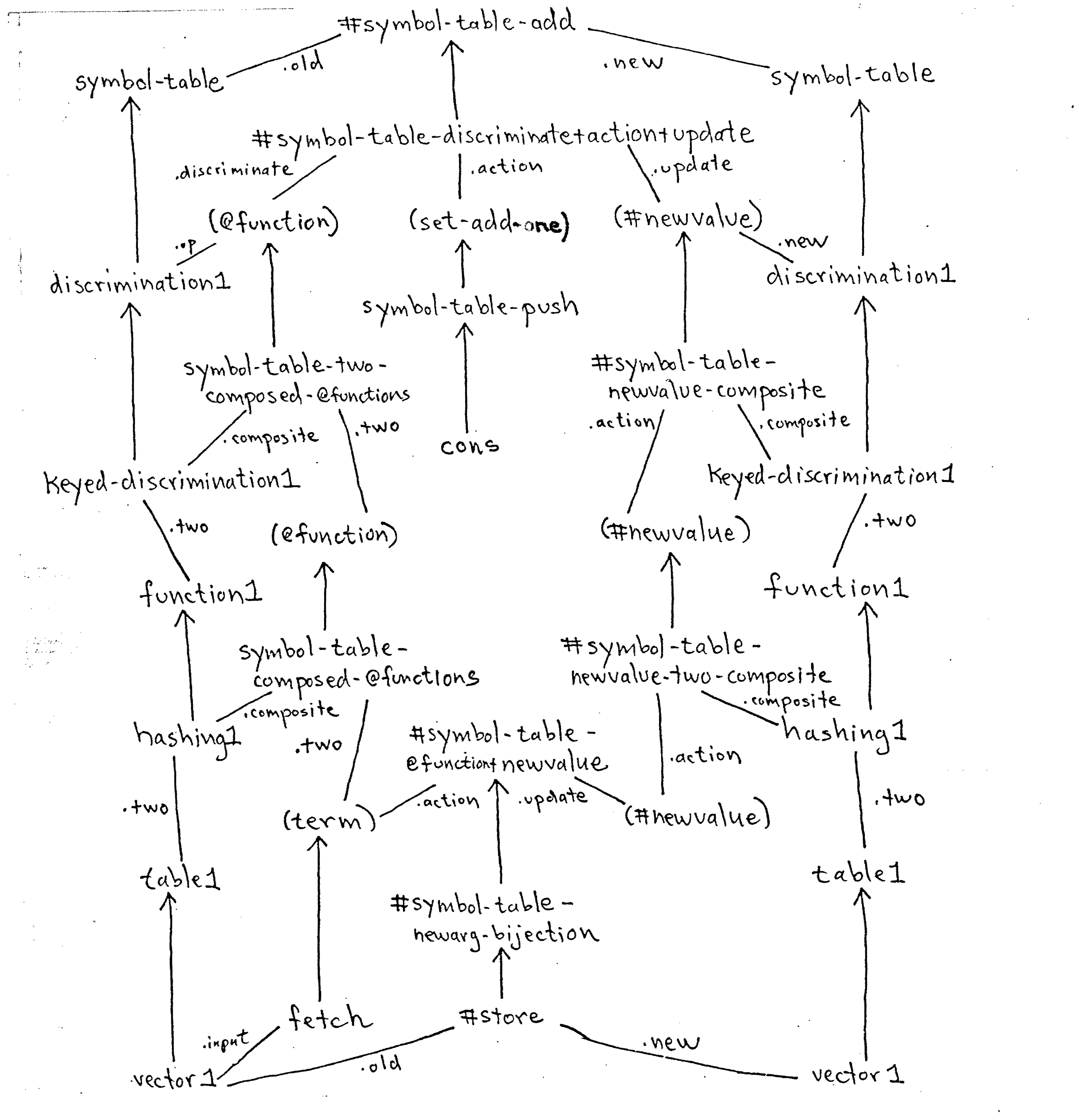


Figure 6-5. Implementation of Addition to Symbol Table.

```
> )
APPLICABLE IMPLEMENTATIONS FOR: #SYMBOL-TABLE-ADD (A SET-ADD-ONE)
      1. PUSH
      2. INTERNAL-THREAD-ADD
      3. DISCRIMINATE+ACTI( N+UPDATE
? 3

LET "#SYMBOL-TABLE-DISCRIMINATE+ACTION+UPDATE" BE THE IMPLEMENTATION OF
#SYMBOL-TABLE-ADD AS DISCRIMINATE+ACTION+UPDATE, SUCH THAT
#SYMBOL-TABLE-DISCRIMINATE+ACTION+UPDATE.ACTION IS SET ADDITION,
AND #SYMBOL-TABLE-DISCRIMINATE+ACTION+UPDATE.UPDATE IS BY SIDE EFFECT.
```

The system begins by retrieving three possible implementations for #Symbol-table-add. The first two are implementations of Set-add for sets implemented as lists or labelled threads; the third overlay (shown in Fig. 6-6) is the implementation of Old+input+output-set (of which Set-add is a specialization) for sets implemented as discriminations. The user chooses the third option,[1] and the system responds as usual by specializing the left hand side plan appropriately.

Notice that the overlay in Fig. 6-6 is between two plans in which no commitment has yet been made as to whether or not side effects are involved. One of the pre-computed properties of this overlay is that if the right hand side is specialized to be by side effect (i.e. #Set-add or #Set-remove), then the Update step on the left hand side is also by side effect (i.e. #Newvalue), and vice versa.

Since there are no overlays for #Symbol-table-discriminate+action+update as a whole, the system looks for implementations of the roles separately. The Discriminate role is an instance of @Function, in which Discrimination1 is the function applied (Op). The further implementation of this role is simply a two level composition of instances of @Function which mirrors the decomposition of Discrimination1 into Symbol, Hash1 and Table1. This is shown in Fig. 6-5, but omitted from the scenario transcript here.

```
> )
APPLICABLE IMPLEMENTATIONS FOR:
              #SYMBOL-TABLE-DISCRIMINATE+ACTION+UPDATE.ACTION (A SET-ADD-ONE)
      1. PUSH
      2. INTERNAL-THREAD-ADD
      3. DISCRIMINATE+ACTION+UPDATE
? 1

LET "SYMBOL-TABLE-PUSH" BE THE IMPLEMENTATION OF
#SYMBOL-TABLE-DISCRIMINATE+ACTION+UPDATE.ACTION AS PUSH.
```

The user chooses to implement Set-add-one operations on the buckets by pushing new elements on the front of the lists.

1. As discussed earlier, if the system assumes the same set is not being implemented two different ways, it could choose this option on its own initiative. However, some clever implementations actually do involve implementing the same abstract data structure simultaneously two different ways.

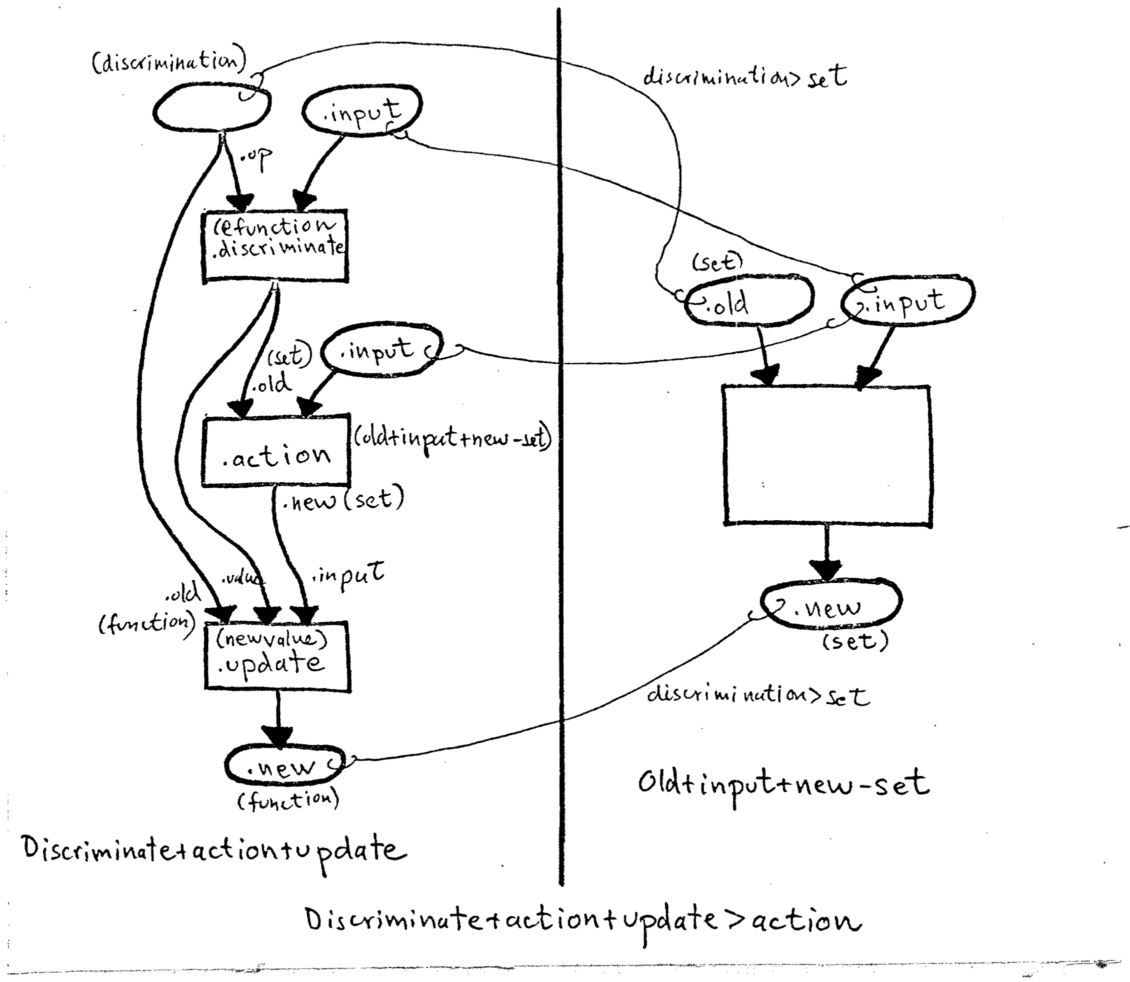


Figure 6-6. Adding and Removing Members from a Discrimination.

```
> ↲
APPLICABLE IMPLEMENTATIONS FOR:
          #SYMBOL-TABLE-DISCRIMINATE+ACTION+UPDATE.UPDATE (A #NEWVALUE)
    1. NEWVALUE-COMPOSITE
? ↲

LET "NEWVALUE-COMPOSITE" BE THE IMPLEMENTATION OF
#SYMBOL-TABLE-DISCRIMINATE+ACTION+UPDATE.UPDATE AS NEWVALUE-COMPOSITE
BY SIDE EFFECT.
```

As discussed in Chapter Four, Newvalue operations on a function implemented as a composition can be implemented by Newvalue operations on the second component only. Furthermore, a property of this implementation is that if the operation on the second component is by side effect, then in effect the composed function has been modified by side effect. In this case, #Newvalue operations on Discrimination1 are implemented as #Newvalue operations on Function1. Similarly (see Fig. 6-5, but not shown here), #Newvalue operations on Function1 are implemented as #Newvalue operations on Table1. This completes implementation of all roles of #Symbol-table-discriminate+action+update.

Prompted by the user, the system continues to suggest overlays for implementing the parts of the design which are not yet down to the level of Lisp primitives. The two simple steps shown below are: (i) to implement Symbol-table-push as CONS (compatible with the implementation of the buckets of the table as Lisp lists), and (ii) to implement Term applied to Table1 as ARRAYFETCH (compatible with the implementation of Table1 as Vector1.)[1]

```
> ↲
APPLICABLE OVERLAYS FOR: SYMBOL-TABLE-PUSH (PUSH)
    1. BUMP+UPDATE
    2. CONS
? 2

> ↲
APPLICABLE OVERLAYS FOR: SYMBOL-TABLE-COMPOSED-@FUNCTIONS.TWO (TERM)
    1. FETCH
? ↲
```

Other simple steps, omitted here, are the implementation of the application of Symbol as CAR, and the application of Hash1 as a procedure call. This leaves only #Newvalue applied to Table1 (see Fig. 6-5) to be implemented further.

```
> ↲
APPLICABLE IMPLEMENTATIONS FOR:
          #SYMBOL-TABLE-NEWVALUE-TWO-COMPOSITE.ACTION (A #NEWVALUE)
    1. NEWVALUE-COMPOSITE
? no

> ↲
NO APPLICABLE OVERLAYS.
```

1. This is skipping the intermediate steps of Table2 and Table3, as discussed earlier.

Unfortunately, the only implementation in the current library for #Newvalue[1] is for a function implemented as a composition of two functions, which is not what we want for Table1. At this point the simple refinement strategy used by the system thus far is stymied. The problem is that in order to implement #Newvalue as the simpler #Newarg (which then becomes ARRAYSTORE for Lisp vectors), the system must recognize that the function involved is one-to-one (a **Bijection**) and that the argument which maps to the old value has already been computed. The plan which the system recognizes is called **@Function+newvalue** and is shown on the right hand side of Fig. 6-7.

The basic idea of the overlay in Fig. 6-7 is that in the special case of one-to-one functions, an instance of @Function followed by Newvalue, as in the Discriminate+action+update plan, can be implemented simply by an instance of Newarg. In other words, if you know that there is only one domain element which maps to a given range element, then updating all domain elements which map to that range element (i.e. Newvalue) degenerates into changing the value associated with that one domain element (i.e. Newarg). Furthermore, in terms of side effects, an impure Update operation (#Newvalue) in @Function+newvalue corresponds to #Newarg.

```
> recognize @function+newvalue.

LET "#SYMBOL-TABLE-@FUNCTION+NEWVALUE" BE A SPECIALIZATION OF
@FUNCTION+NEWVALUE SUCH THAT
#SYMBOL-TABLE-@FUNCTION+NEWVALUE.ACTION.OP=TABLE1 .
```

The user guides the system at this point by advising it to try to recognize an instance of the plan @Function+newvalue somewhere in the current design. Given the focus of trying to recognize only one particular plan, the system succeeds in noticing that Symbol-table-composed-@functions.Two (see Fig. 6-5) together with the #Symbol-table-newvalue-two-composite.Action satisfy the constraints of @Function+newvalue.[2] What has happened here is that parts of two different branches of the tree have been grouped together to recognize a plan which has a known implementation. This is a novel feature of this synthesis scenario as compared to the standard top-down refinement approach.

```
> ⤶
APPLICABLE IMPLEMENTATIONS FOR:
          #SYMBOL-TABLE-@FUNCTION+NEWVALUE (A @FUNCTION+NEWVALUE)
     1. NEWARG-BIJECTION
? ⤶

LET "#SYMBOL-TABLE-NEWARG-BIJECTION" BE THE IMPLEMENTATION OF
#SYMBOL-TABLE-@FUNCTION+NEWVALUE AS NEWARG-BIJECTION.
```

Now the overlay can be applied which implements #Newvalue as #Newarg, or in the case of a sequence, #Newterm, and finally, as #Store.

---

1. Recall that Newvalue is the specification for updating a function such that all arguments that used to map to a given value, map to a new given value.
2. Table1 is an irredundant sequence, which means it is a one-to-one function.

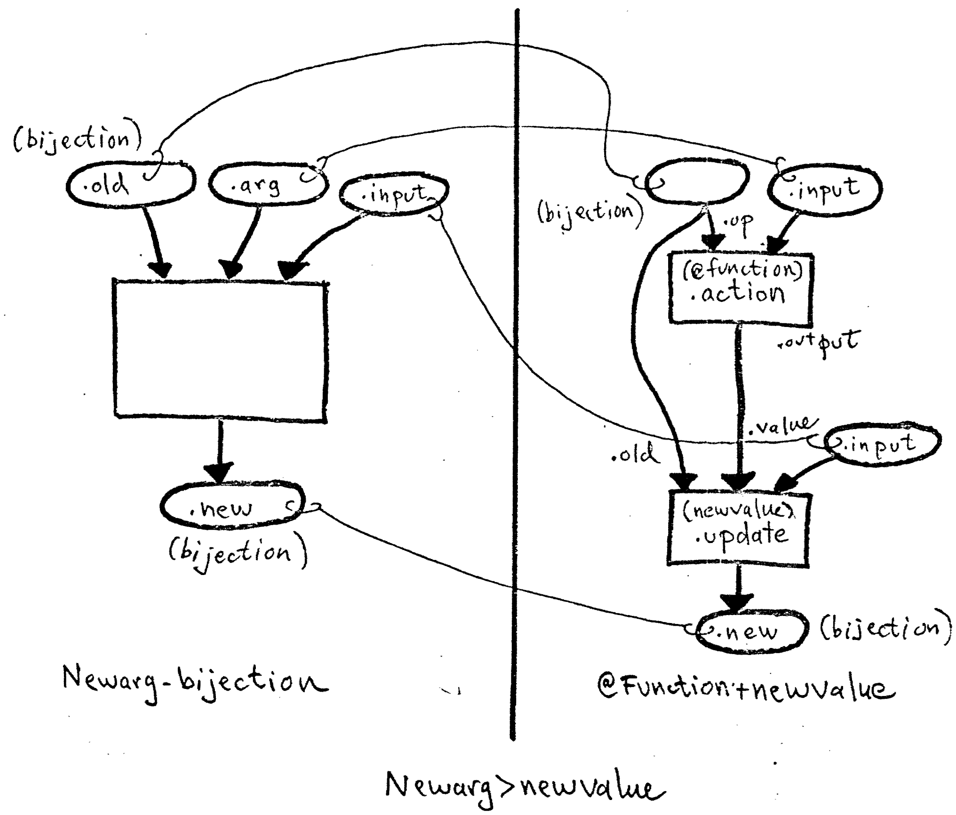


Figure 6-7. Updating a Bijection.

```
> ↵
APPLICABLE IMPLEMENTATIONS FOR: #SYMBOL-TABLE-NEWARG-BIJECTION (A #NEWTERM)
      1. #STORE
? ↵
```

Code generation follows in a similar fashion to before.

```
> show code for #symbol-table-add

      (DEFINE SYMBOL-TABLE-ADD
         (LAMBDA (TABLE INPUT)                 ;MODIFIES TABLE.
            (PROG (INDEX)
                  (SETQ INDEX (HASH (CAR INPUT)))
                  (ARRAYSTORE TABLE INDEX
                              (CONS (ARRAYFETCH TABLE INDEX)
                                    INPUT)))))
```

Synthesis of Associative Deletion

The last procedure to be synthesized is for associative deletion of entries in the symbol table. This procedure and its development share many features with the retrieval and addition procedures, which have already been presented in detail. This part of the scenario will therefore be brief and will for the most part rely on Fig. 6-8 rather than showing all of the system-user interactions, as in the preceding sections.

```
> let "symbol table expunge" be a specialization of expunge by side effect such
  that the old set is a symbol table, the key function is symbol,
  and there exists a unique "x" such that x belongs to the old set
  and the key function applied to x equals the input.
```

These are the starting specifications. **Expunge** is a standard input-output specification in the library for deleting from a set on the basis of a given key value. The deletion here is by side effect, and there is an additional precondition specified, namely that there is exactly one entry in the table with the given key. This precondition specializes Expunge to **Expunge-one**, a standard specialization of Expunge in the library.

```
> ↵
APPLICABLE IMPLEMENTATIONS FOR: #SYMBOL-TABLE-EXPUNGE (AN EXPUNGE-ONE)
      1. RESTRICT-COMPOSITE
      2. KEYED-DISCRIMINATE+EXPUNGE+UPDATE
? 2

LET "#SYMBOL-TABLE-KEYED-DISCRIMINATE+EXPUNGE+UPDATE" BE THE IMPLEMENTATION
OF #SYMBOL-TABLE-EXPUNGE AS KEYED-DISCRIMINATE+EXPUNGE+UPDATE.
```

For sets implemented as keyed discriminations, Expunge is implemented by the three step plan Discriminate+expunge+update, shown in Fig. 6-9, which is similar to Discriminate+action+update in the implementation of SYMBOL-TABLE-ADD. Like Discriminate+action+update the pre-compiled side effect analysis of this plan says that the side effect implementation is achieve by specializing the Update step to #Newvalue. Part of the cleverness in this synthesis example involves avoiding the Update step entirely by performing the Action by side effect instead.

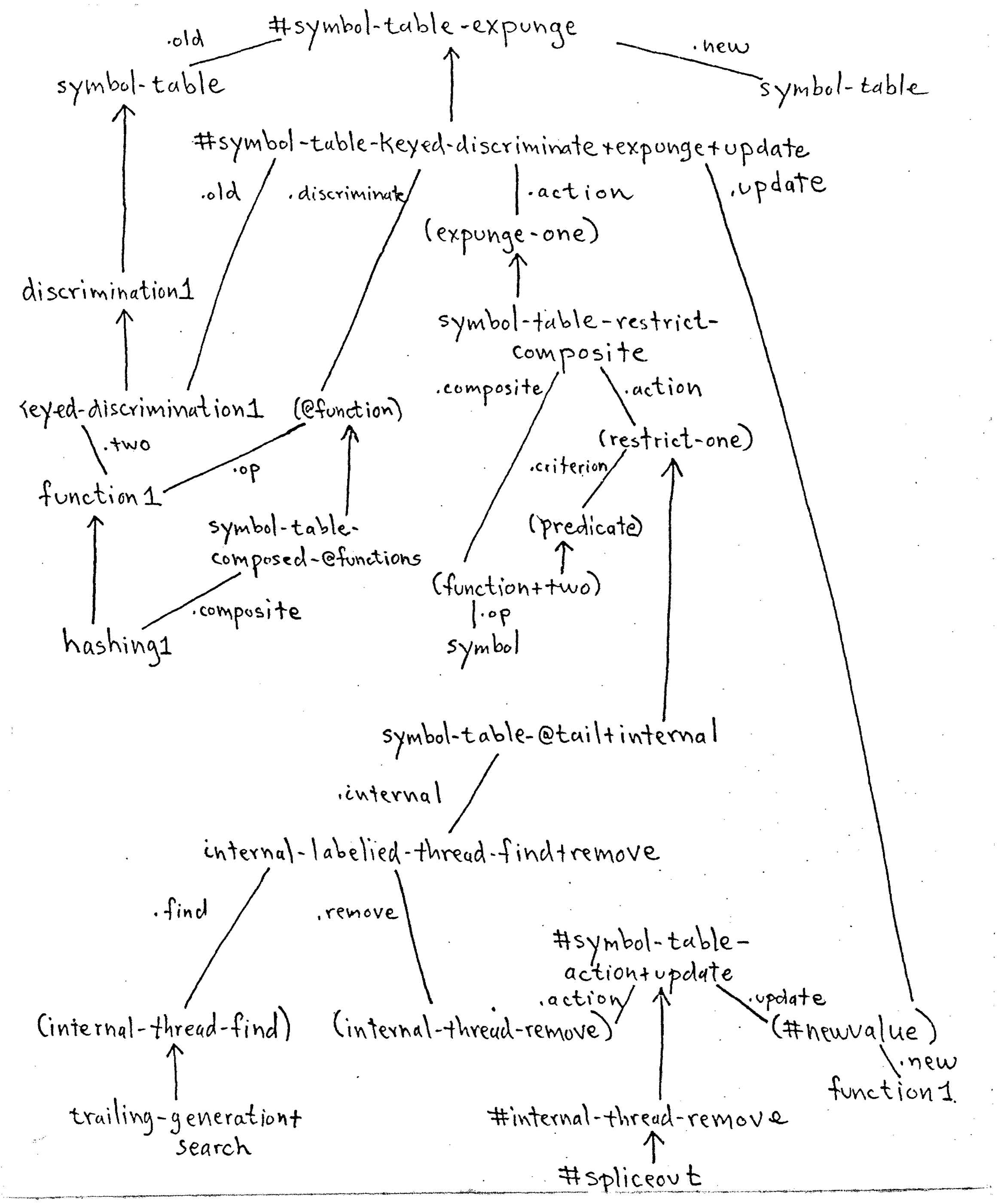


Figure 6-8. Implementation of Associative Deletion from Symbol Table.

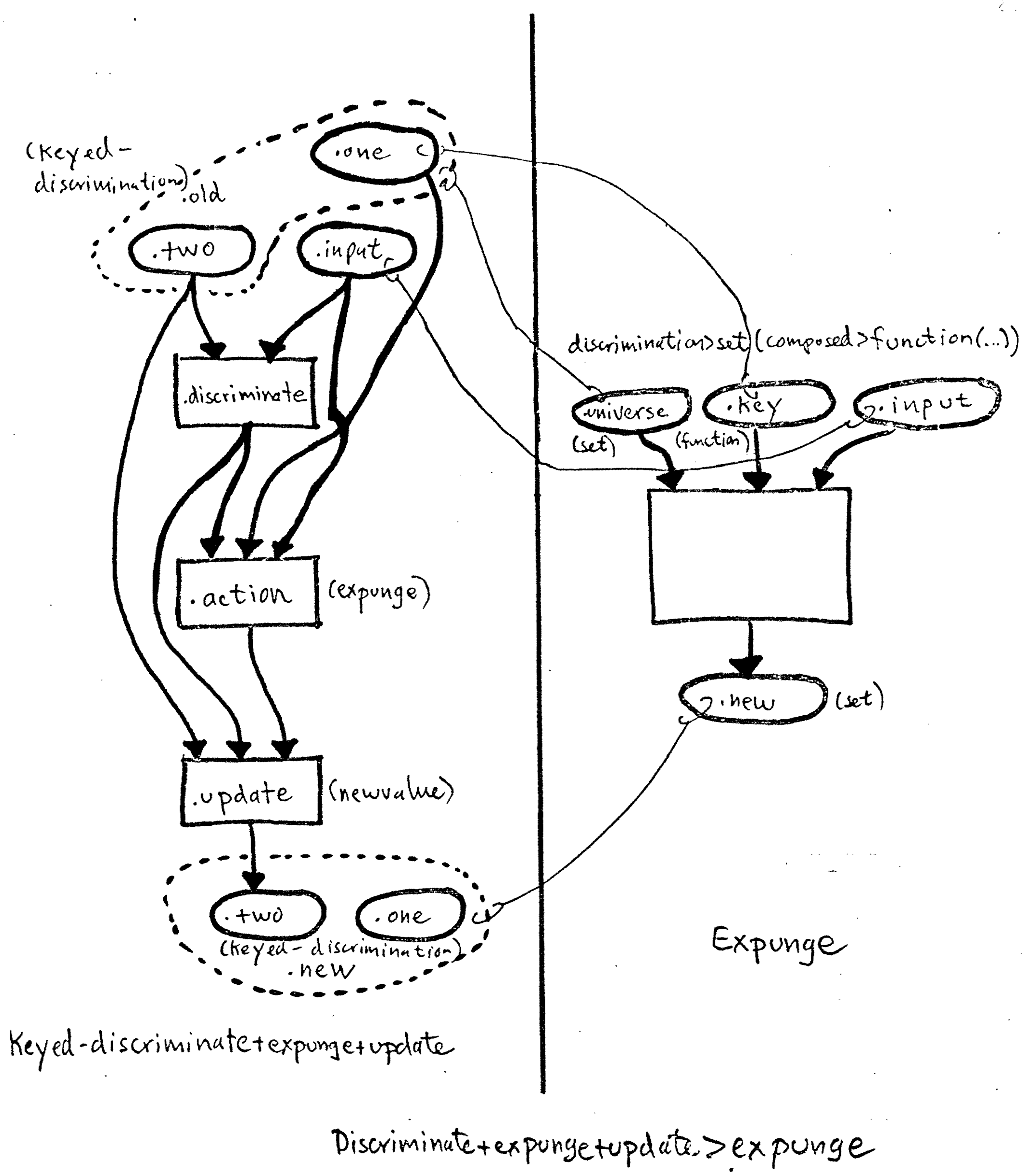


Figure 6-9. Associative Deletion from Keyed Discrimination.

The first step in Discriminate+expunge+update, Discriminate, is an instance of @Function which computes the appropriate bucket from the given key. The implementation of this step uses the plan Symbol-table-composed-@functions, which was developed in the synthesis of SYMBOL-TABLE-RETRIEVE (see Fig. 6-8).

```
> )
 APPLICABLE IMPLEMENTATIONS FOR:
         #SYMBOL-TABLE-KEYED-DISCRIMINATE+EXPUNGE+UPDATE.ACTION (AN EXPUNGE-ONE)
     1. RESTRICT-COMPOSITE
     2. KEYED-DISCRIMINATE+EXPUNGE+UPDATE
? 1

LET "SYMBOL-TABLE-RESTRICT-COMPOSITE" BE THE IMPLEMENTATION OF
#SYMBOL-TABLE-KEYED-DISCRIMINATE+EXPUNGE+UPDATE.ACTION AS RESTRICT-COMPOSITE.
```

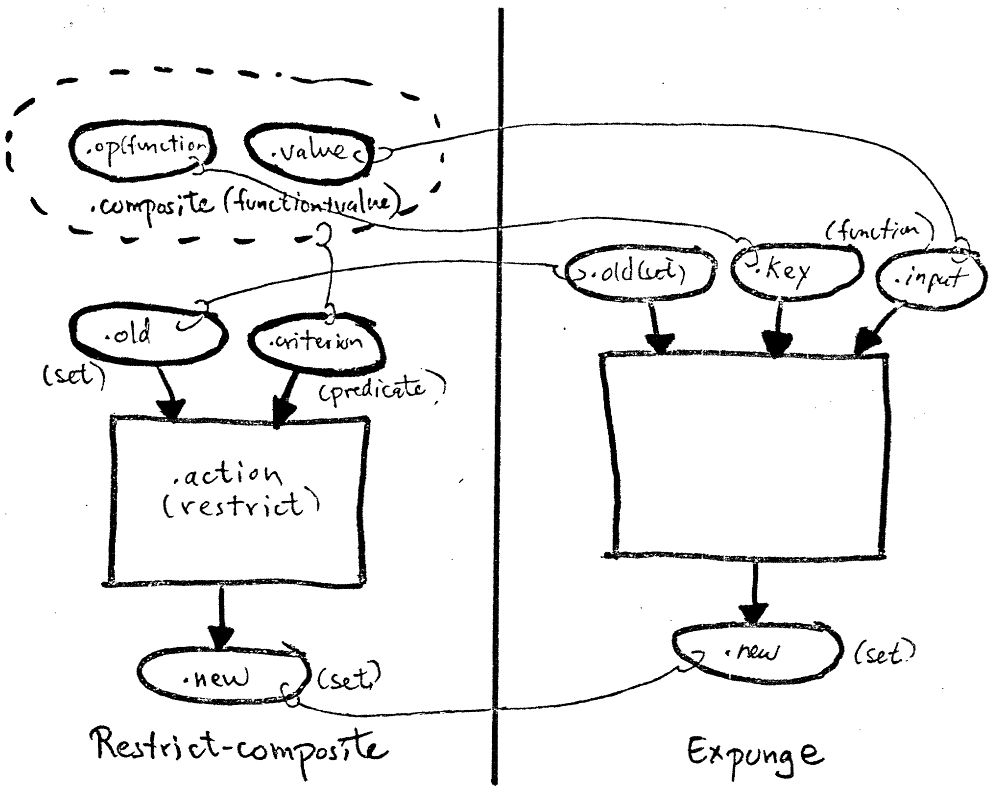
The Expunge-one action on the buckets is implemented in the default way using Restrict, in which the criterion is a composition of the Symbol function and #Symbol-table-expunge.Key. This overlay is shown in Fig. 6-10. It is similar to the implementation of Retrieve as Any-composite in SYMBOL-TABLE-RETRIEVE. Furthermore, it is a property of this overlay that if the right hand side is specialized to Expunge-one, then the Action on the left hand side is correspondingly specialized to **Restrict-one**, in which there is expected to be only one member of the Universe set which satisfies the given Criterion.

```
> )
APPLICABLE IMPLEMENTATIONS FOR:
             SYMBOL-TABLE-RESTRICT-COMPOSITE.ACTION (A RESTRICT-ONE)
     1. ITERATIVE-FILTERING
     2. @TAIL+INTERNAL
? 2

LET "SYMBOL-TABLE-@TAIL+INTERNAL" BE THE IMPLEMENTATION OF
SYMBOL-TABLE-RESTRICT-COMPOSITE.ACTION AS @TAIL+INTERNAL.
```

Restrict can be implemented either as a filtering loop, or as the plan **@Tail+internal**, shown in Fig. 6-11. This plan removes a member from a set implemented as an irredundant list.

Removing a member from a set implemented as an irredundant list breaks down into two cases: if it happens that the member to be removed is the head of the list, then removal is achieved simply by a taking the tail of the list; otherwise, viewing the list as a labelled thread, the internal node of the spine which is labelled with the given member must be found and removed. These two cases will eventually manifest themselves in the code as follows:



Restrict>expunge

Figure 6-10. Default Implementation of Associative Deletion.

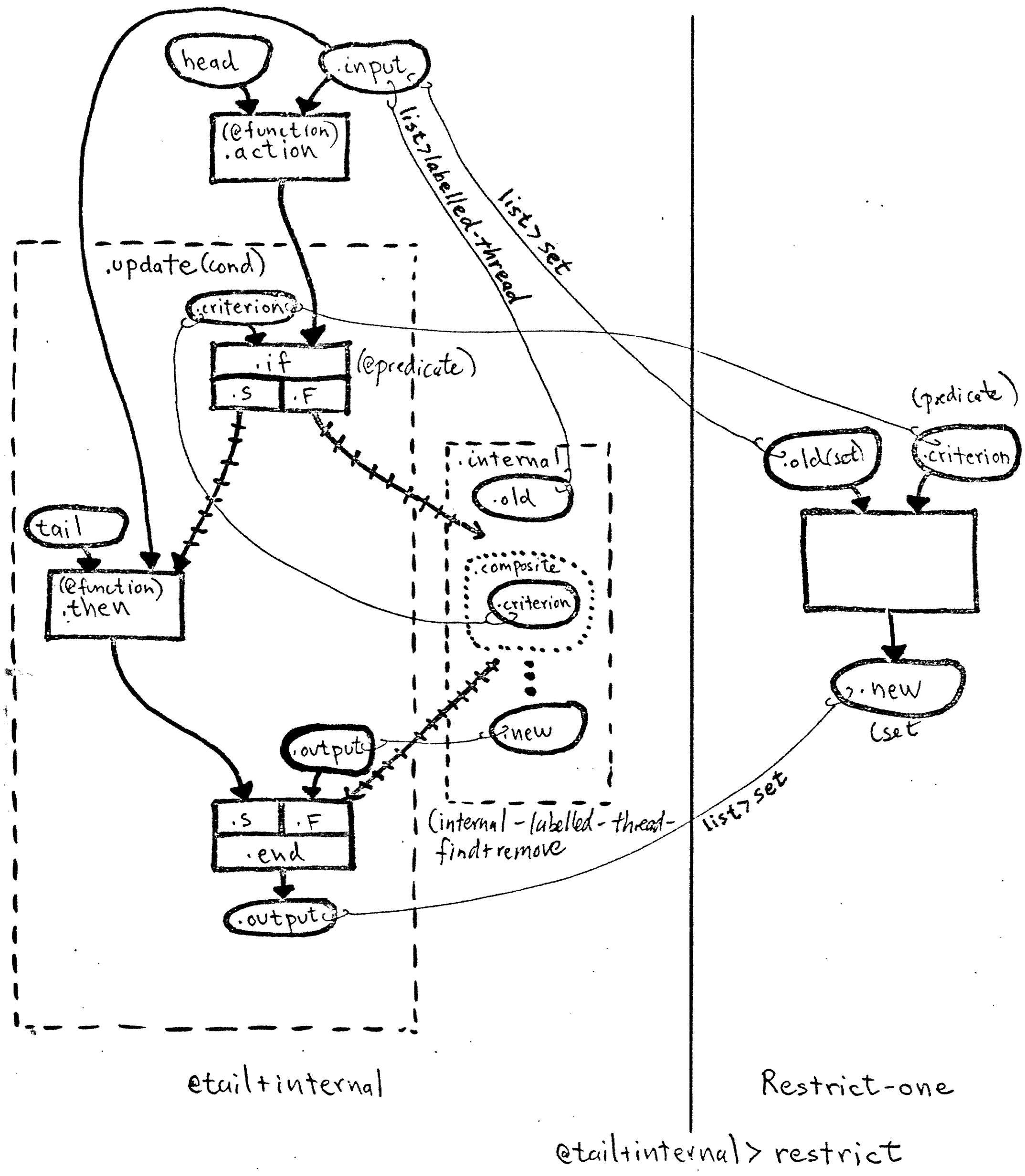


Figure 6-11. Set Removal for Irredundant Lists.

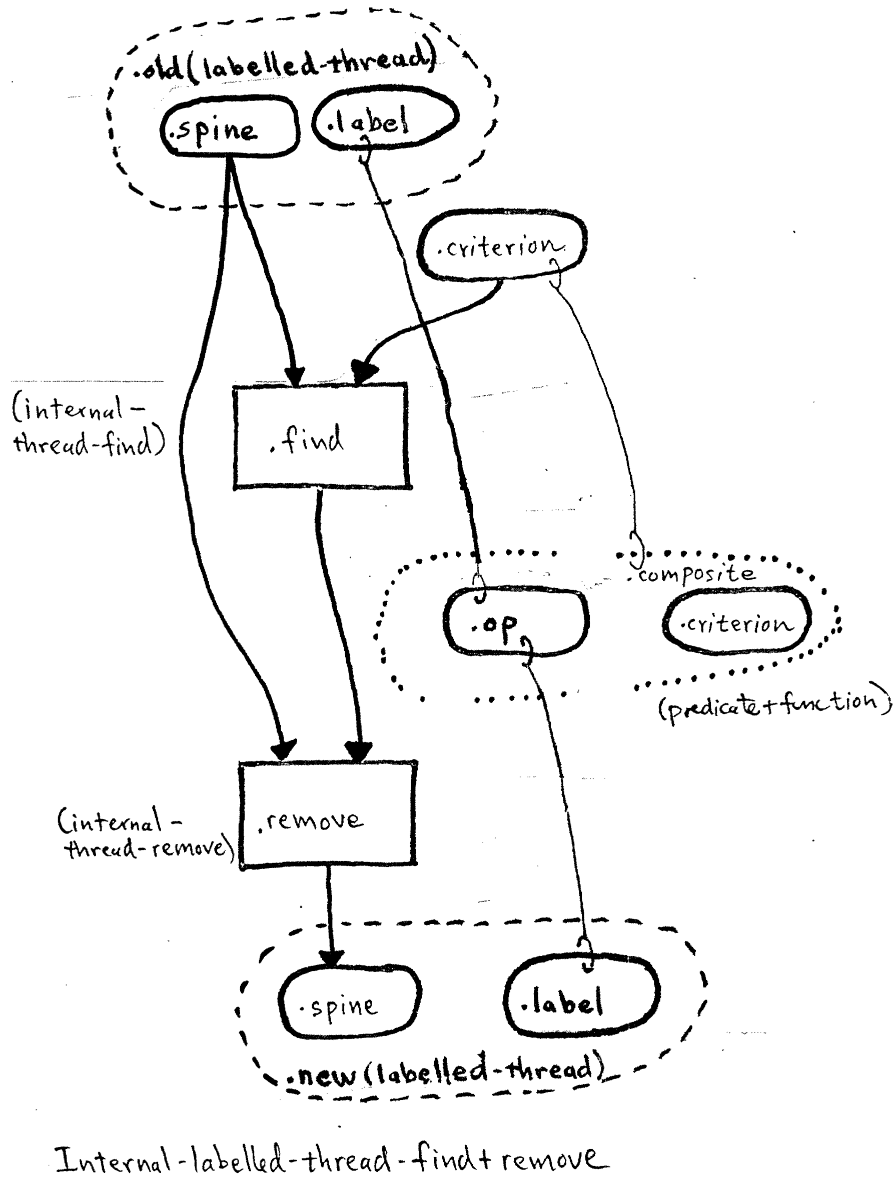


Figure 6-12. Internal Labelled Thread Find and Remove.

```
(DEFINE SYMBOL-TABLE-EXPUNGE
   (LAMBDA (... INPUT)
      (PROG (... BUCKET PREVIOUS)
            (SETQ PREVIOUS ...)
            (COND ((EQ (CAAR PREVIOUS) INPUT)
                   (... (CDR PREVIOUS))
                   (RETURN NIL)))
         LP ...
            (COND ((EQ (CAAR BUCKET) INPUT)
                   (RPLACD PREVIOUS (CDR BUCKET))
                   (RETURN NIL)))
            ...
            (GO LP))))
```

Let us first consider the overlay @Tail+internal>restrict in Fig. 6-11, which formalizes the breakdown into two cases described above. On the right hand side of this overlay we have Restrict-one, which specifies the removal of the (unique) member of a set which does not satisfy a given criterion. The top level structure of the plan on the left, which implements these specifications, is a conditional (Cond). The Input to the test of this conditional is the head of the irredundant list which implements the Old set; the criterion is the complement of the criterion of Restrict-one. The output of this conditional (End.output) is the irredundant list which implements the New set. In the Succeed case (i.e. when the head of the input list satisfies the given criterion), this output is the resulting of taking the tail of the input list. In the Fail case, the new list is computed by Internal-labelled-thread-find+remove.

Internal-labelled-thread-find+remove, shown in Fig. 6-12, is an extension of Internal-thread-find+remove. In this plan, the old and new lists are thought of as labelled threads. Internal-labelled-thread-find+remove removes an internal node from the spine of a labelled thread (Old), the label of which satisfies a given predicate, resulting in a (New) labelled thread. As used in @Tail+internal, the criterion applied by Find to each node in the spine of the labelled list is composed from Update.If.Criterion and the label function of the list viewed as a labelled thread, according to the overlay Predicate+function>predicate, given in the appendix. The basic idea of this construction is to test the label of each node, rather than the node itself. Thus for example, if the label function is Car (as in the case of Lisp lists), and Update.If.Criterion is P, then the criterion of the Find step is Q defined as follows:

$$Q(x) \equiv P(Car(x))$$

```
> )
  APPLICABLE IMPLEMENTATIONS FOR:
                 SYMBOL-TABLE-@TAIL+INTERNAL.INTERNAL.FIND (AN INTERNAL-THREAD-FIND)
        1. TRAILING-GENERATION+SEARCH
  ? )
```

The Find role of Internal-labelled-thread-find+remove, which is an instance of Internal-thread-find, is implemented as a Trailing-generation+search loop, as shown in Fig. 6-13. The Universe of Internal-thread-find is the thread generated by the trailing generation, and the two outputs of the loop correspond to the two outputs of Internal-thread-find. This plan will eventually appear in the code as follows, in which the function being applied by the action is Cdr, the Current object is in BUCKET and the Previous object in PREVIOUS.

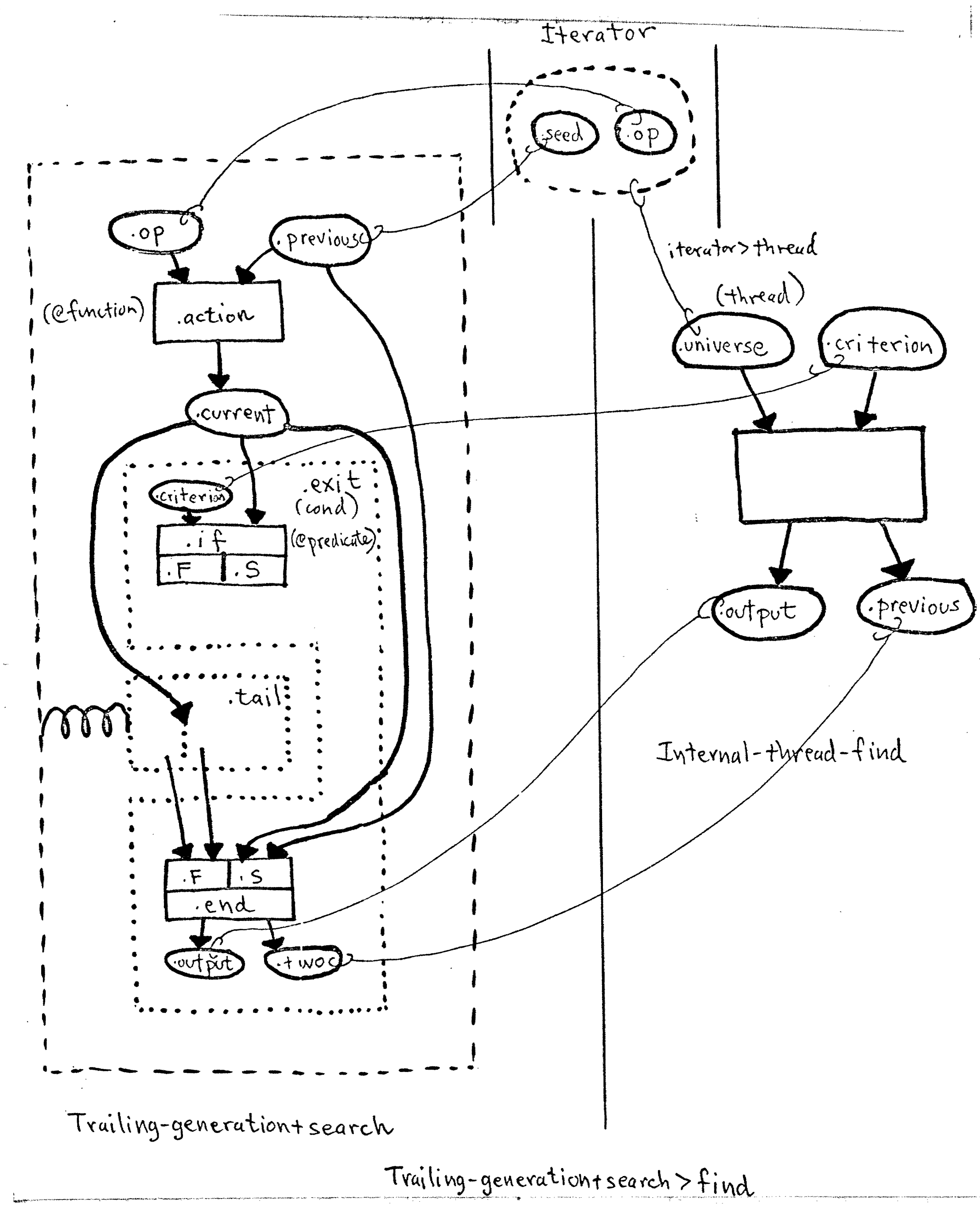


Figure 6-13. Loop to Find Location in Thread.

```
(PROG (... BUCKET PREVIOUS)
      (SETQ PREVIOUS ...)
   LP (SETQ BUCKET (CDR PREVIOUS))
      (COND ((...BUCKET...)
             ...PREVIOUS...BUCKET...
             (RETURN NIL)))
      (SETQ PREVIOUS BUCKET)
      (GO LP))
```

The system then proposes to implement Internal-thread-remove by splicing out, but the user intervenes to suggest a reanalysis.

```
> )
  APPLICABLE IMPLEMENTATIONS FOR: SYMBOL-TABLE-@TAIL+INTERNAL.INTERNAL.REMOVE
                                           (AN INTERNAL-THREAD-REMOVE)
        1. SPLICEOUT
  ? no

> recognize #action+update.

  LET #SYMBOL-TABLE-ACTION+UPDATE BE A SPECIALIZATION OF #ACTION+UPDATE
  SUCH THAT #SYMBOL-TABLE-ACTION+UPDATE.UPDATE.OLD=FUNCTION1 .
```

In contrast to SYMBOL-TABLE-ADD, where advice from the user was crucial to completing the synthesis, this intervention is merely to cause the system to come out with a more efficient program. In particular we want the system to realize that, if Internal-thread-remove is implemented by side effect, then when the member of the bucket to be deleted is not the Head, the operation to update the table is not necessary. This piece of implementation knowledge is represented in the library by the overlay **#Old+input+new>action+update**, which will be discussed further in Chapter Eight. The basic idea of this overlay, however, is that in general, modifying a range element amounts to modifying the function. In order to apply this overlay, however, the system must first group together parts of plans on different branches of the tree (see Fig. 6-8), as was the case in the synthesis of SYMBOL-TABLE-ADD.

Thus the system implements the Internal.Remove step of Symbol-table-@tail+internal as #Internal-thread-remove, which is further implemented as #Spliceout, as shown in Fig. 6-14.

Spliceout has four roles: Old and New, which are iterators with the same seed; Bump, which is an instance of Apply; and Splice, which is an instance of Newarg. The purpose of Bump is to get the successor of the node to be removed, which becomes the Input of Splice. The Arg of Splice is the predecessor of the Input of Bump (which typically comes from an instance of Internal-thread-find). The Op of the Old iterator (e.g. Cdr for Lisp lists) is the Op input to both Bump and Splice; the Op of the new iterator is the output of Splice. This plan will eventually emerge as the following code.

```
(RPLACD PREVIOUS (CDR BUCKET))
```

Spliceout implements Internal-thread-remove as described by the overlay **Spliceout>remove**, shown in Fig. 6-14. The old iterator implements the old thread, and the new iterator implements the new thread. The node being deleted is the Input of Bump. Notice that the Arg input to Splice in the Spliceout plan (the predecessor of the node deleted) has no corresponding object on the right hand side of the overlay. This means that as far as this overlay is concerned, some other part of the program surrounding an

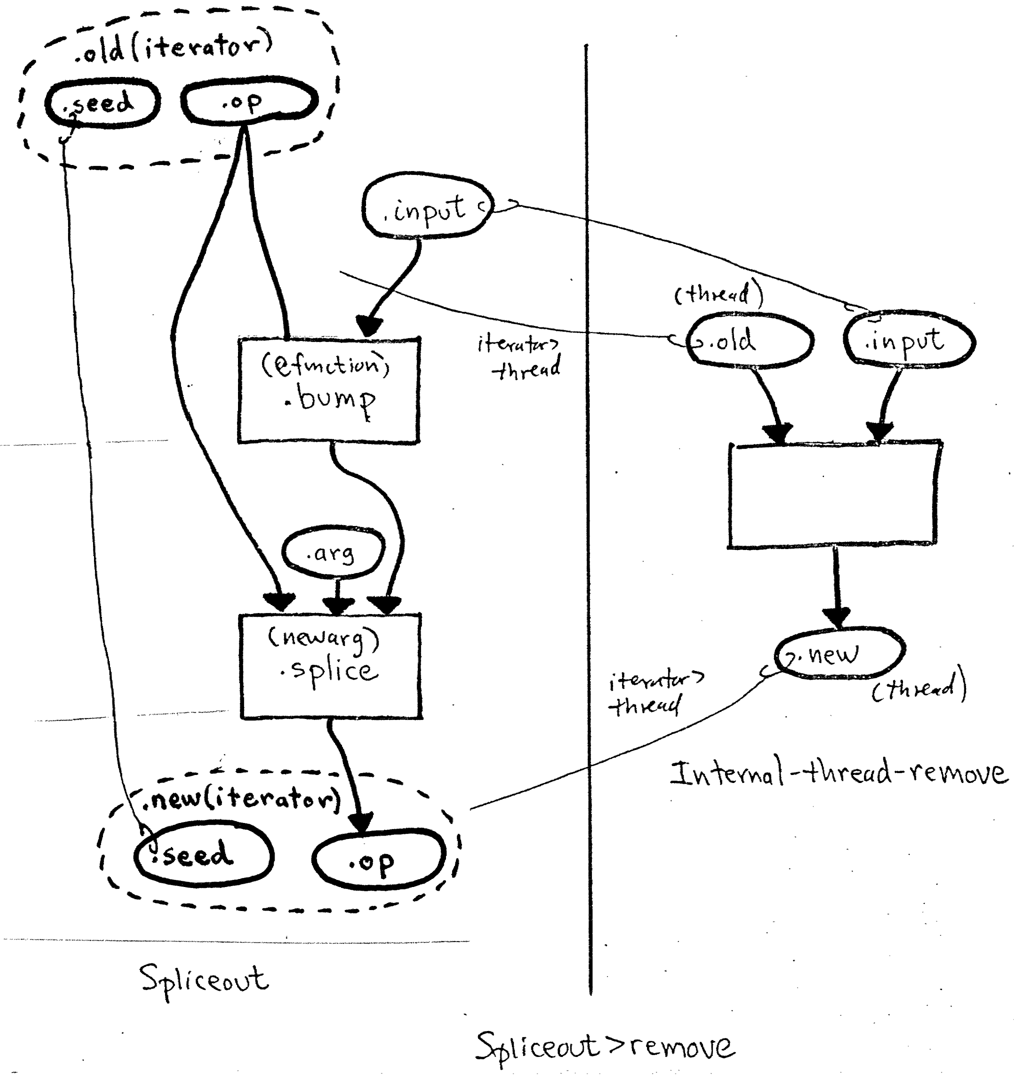


Figure 6-14. Removing from a Thread by Splicing Out

instance of the left hand side (e.g. the Internal-thread-find) must provide an Arg input to Splice which satisfies the successor constraint. In other words this is an implementation of Internal-thread-remove for the case when we already know the location of the node to be removed.

By the further rearrangement and straightforward implementation steps, we arrive finally at a surface plan which can then be turned over to the code generator. The resulting code is essentially the same as in the scenario of Chapter Two.

```
> show code for #symbol-table-expunge.

        (DEFINE SYMBOL-TABLE-EXPUNGE
           (LAMBDA (TABLE INPUT)
              (PROG (INDEX BUCKET PREVIOUS)
                    (SETQ INDEX (HASH INPUT))
                    (SETQ PREVIOUS (ARRAYFETCH TABLE INDEX))
                    (COND ((EQ (CAAR PREVIOUS) INPUT)
                           (ARRAYSTORE TABLE INDEX (CDR PREVIOUS))
                           (RETURN NIL)))
                 LP (SETQ BUCKET (CDR PREVIOUS))
                    (COND ((EQ (CAAR BUCKET) INPUT)
                           (RPLACD PREVIOUS (CDR BUCKET))
                           (RETURN NIL)))
                    (SETQ PREVIOUS BUCKET)
                    (GO LP))))
```

CHAPTER SEVEN

VERIFICATION BY INSPECTION

This brief chapter outlines the applicability of inspection methods and the plan library to program verification. Program verification has at least two main aspects:

(i) increasing confidence in the correctness of a program,

(ii) detecting potential errors.

Verifying Overlays

The use of the plan library can increase confidence in the correctness of a program by virtue of the fact that overlays in the library can be pre-verified. If a program is constructed entirely out of plans and overlays from the library, then it is guaranteed to be correct (in the sense of there being no implementation errors — the program may still not do what the programmer wanted in the ultimate sense). To the extent that parts of a program are constructed using the library, confidence in the correctness of the program is increased.

A completely formal statement of the correctness conditions on overlays depends on the logical foundations of the plan calculus developed in Chapter Eight. The basic idea, however, is to verify that the function defined by an overlay and its inverse mapping are both *total*, i.e. they are defined on all instances of the left and right hand hand side plans, respectively. Practically speaking, the effect of this definition of correctness is to force all of the conditions required for the correct use of an overlay to be explicitly stated in the constraints of the plans on both sides.

Note that techniques for automatically verifying the correctness of overlays are not the concern of this report. The important point established here is only that there exists for the plan calculus a formally definable and usable notion of correctness, which has not been the case for other formalisms used to represent the same knowledge. Given the formal definitions in Chapter Eight, it is possible to verify overlays to whatever degree of rigor is warranted, up to and including a step-by-step formal proof in first order logic (which might be mechanically produced). Note however that these proofs can be quite difficult and idiosyncratic, depending as they do on the mathematical properties of the various programming domains involved; but this is as one would expect. The thrust of using inspection methods is to take advantage of this effort by re-using these proofs as lemmas.

Near-Miss Recognition

An inspection method for error detection is *near-miss* recognition. In near-miss recognition, most but not all of the constraints of a plan are satisfied. If part of a user's design almost matches a plan in the library, the discrepancy between the two descriptions can be brought to the user's attention as a potential error. This method of error detection makes use of the correct plans in the library to detect errors, rather than explicitly adding a taxonomy of errors to the "grammar" as in Ruth [59].

Like all inspection methods, error detection by inspection is not as powerful as more general methods. However, it has the advantage that potential errors are characterized in terms which are closer to the engineering vocabulary of the user's design. The remainder of this chapter gives an example in detail. The method described in this example has not yet been implemented, however the implementation of an algorithm for near-miss pattern matching using the plan library is currently in progress by Brotsky [8].

In the scenario of Chapter Two, the user typed in the following code for finding an element in a list satisfying a given criterion, and splicing it out.

```
(DEFINE BUCKET-DELETE
   (LAMBDA (BUCKET INPUT)                ;MODIFIES BUCKET.
      (PROG (P Q)
            (SETQ P BUCKET)
         LP (COND ((EQUAL (CAAR P) INPUT)
                   (RPLACD Q P)          ;SPLICE OUT.
                   (RETURN BUCKET)))
            (SETQ Q P)
            (SETQ P (CDR P))
            (GO LP))))
```

There were two errors detected in this code: one in the loop that finds the element, and one in the splicing out. This section discusses only the detection of the first error.

The first step in detecting an error is to translate the code above into the plan calculus as discussed in Chapters Four and Five. The surface plan for the loop part of BUCKET-DELETE (not including the splice out after the loop) is shown on the left of Fig. 7-1. To make this example easier to follow, the surface plan shown in the figure has been simplified by omitting the control flow arcs (since the important recognition in this example depends on the data flow), and by assuming that the code (EQUAL (CAAR P) INPUT) has already been analyzed as the testing of P by a composite predicate made up out of the Eq relation, the Caar function and INPUT.[1]

The next steps are to recognize Trailing-search and Iterative-generation in the surface plan for BUCKET-DELETE. Fig. 7-1 illustrates the recognition of Trailing-search. **Trailing-search**, shown on the right hand side of the figure, is a loop plan with four roles: Exit, Tail, Current and Previous. As in all loop plans,[2] the recursively defined role is called Tail. The Exit role is a conditional plan which groups together the exit test (Exit.If) of the loop and the join (Exit.End) "on the way up". If the exit test succeeds, the loop terminates (and the input to the test is available through the join as an output of the loop); otherwise it continues. The Current and Previous roles are what make this plan a *trailing* loop. The Current object on each iteration is the same as the Previous object on the next iteration (Tail.Previous). The Current object in a trailing search loop is the input to the test; and both the Current and the Previous object are available through the join (Exit.End) as outputs of the loop. Exit.End is an

1. Using the overlays Binrel+two>predicate and Predicate+function>predicate (see appendix).
2. A detailed taxonomy of loop plans is given in Chapter Nine.

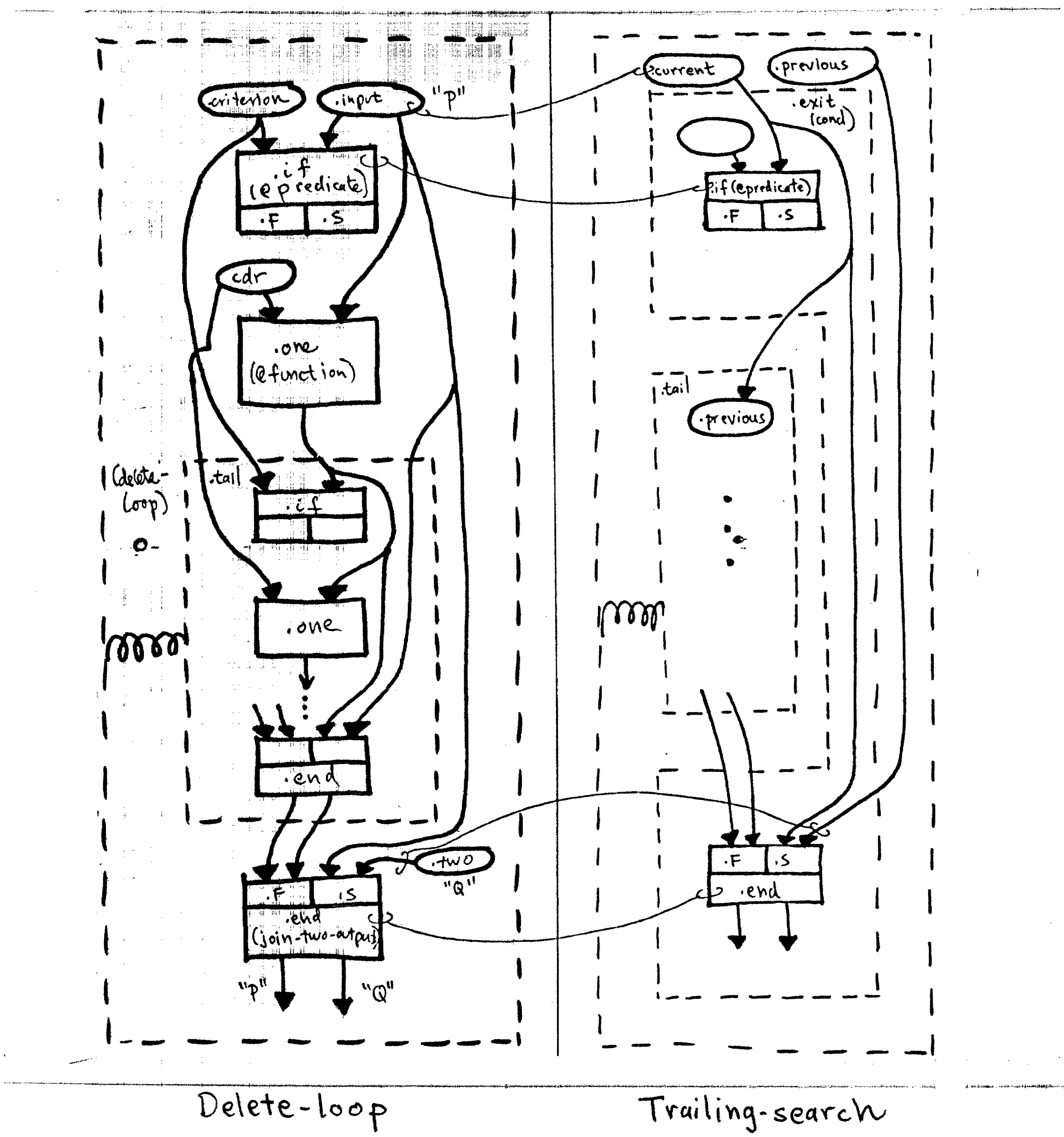


Figure 7-1. Recognizing Trailing Search in Bucket-Delete.

instance of **Join-two-outputs**, an extension of Join-outputs (see Chapter Eight) in which two outputs are joined.

The surface plan for BUCKET-DELETE, Delete-loop, can be analyzed as Trailing-search by identifying Delete-loop.If with Trailing-search.Exit.If (in which case P in the code holds to the Current object), and identifying Delete-loop.End with Trailing-search.Exit.End (in which case Q in the code holds to the Previous object).

Fig. 7-2 illustrates the recognition of Iterative-generation in the surface plan for BUCKET-DELETE. **Iterative-generation** is the plan for repeatedly applying a given function (the same function each time) to the output of the preceding application of that function. This plan has two roles: Action and Tail. Action is an instance of @Function, in which the function is applied; Tail is the standard recursive invocation. Delete-loop can be analyzed as Iterative-generation by identifying Delete-loop.One with Iterative-generation.Action, as shown in the figure.

Following the recognition of Trailing-search and Iterative-generation, the system also checks whether any standard specializations or extensions of these plans are applicable. In this example, the system finds **Trailing-generation+search** in the library, which is an extension of both Trailing-search and Iterative-generation. Trailing-generation+search has five roles: Exit, Current and Previous (inherited from Trailing-search); Action (inherited from Iterative-generation); and Tail (recursively defined as in both Trailing-search and Iterative-generation). Trailing-generation+search also inherits all the constraints between these roles from both Trailing-search and Iterative-generation, and adds one more, a data flow constraint between Action.Output and Exit.If.Input.

When the system tries to recognize Trailing-generation+search in Delete-loop, it finds that only one constraint is missing — the data flow from Action.Output to Exit.If.Input. Furthermore, this is taken to be a near-miss, rather than a simple failure to match,[1] and the following message is generated warning the programmer about a potential error.

```
WARNING! THE LOOP IN BUCKET-DELETE IS ALMOST A
TRAILING GENERATION AND SEARCH,
        CURRENT: P
        PREVIOUS: Q
        EXIT: (COND ((EQUAL (CAAR P) ...)))
        ACTION: (CDR P)
EXCEPT THAT THE OUTPUT OF THE ACTION IS NOT EQUAL TO THE
INPUT OF THE EXIT TEST.
```

Notice that because this warning message is generated as the result of a near-miss recognition, the programmer gets much more contextual information than would result from detecting this error by other means (e.g. by noticing that the variable Q could be used before it is set). In particular, the system is able to identify the roles played by the parts of the program, i.e. P and Q are not just any variables in the program, but are the current and previous values of a trailing search, and so on. This information makes it easier for the programmer to correct the problem.

1. The exact criteria for distinguishing in general between near-misses and failures to match will have to be determined experimentally.

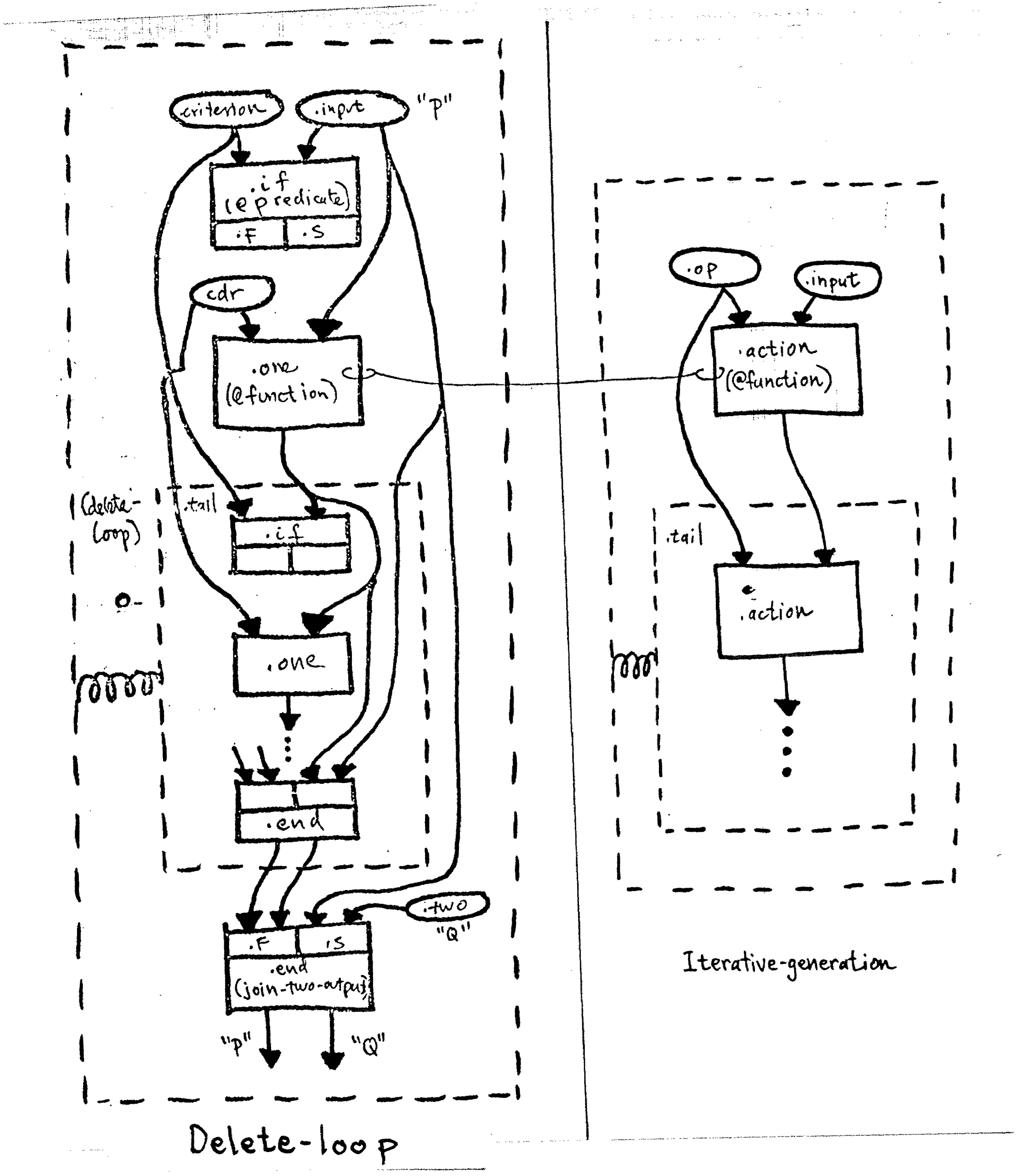


Figure 7-2. Recognition of Iterative-generation in Bucket-Delete.

CHAPTER EIGHT

LOGICAL FOUNDATIONS

8.1 Introduction

The preceding chapters have focused on how the plan calculus can be used in a program understanding system, such as a programmer's apprentice. This chapter takes a more semantic and formal approach to plans. We begin by defining a logical language, similar to the situational calculus used by Green [31] and McCarthy and Hayes [44], which is adequate for expressing the fundamental computational concepts underlying the plan calculus.

We then use the situational calculus to provide a semantic foundation for the plan calculus by giving rules for translating plans into sets of axioms in the situational calculus. The presentation of these rules will be done in two stages. First we will develop enough of the situational calculus to support the semantics of data plans and data overlays. We will then add a notion of temporal order and give the translation rules for temporal plans and temporal overlays.

One important reason for providing a formal semantics for the plan calculus is in order to state precisely the rules of inference on plans. These rules of inference provide the answers to questions such as whether one plan subsumes another, and whether one plan is a correct implementation of another. This is particularly important in order to pre-verify plans in the plan library.

In most of this chapter, examples of Lisp computations will be used to motivate various aspects of the formalism. However, the logical framework developed here is equally applicable to other conventional sequential programming languages.

8.2 Mutable Objects and Side Effects

The everyday world of physical objects is a system with mutable objects and side effects. For example, if I drill a hole in my dining room table, I normally choose to think of it as the same object even though it now behaves differently (i.e. has different properties). Similarly for a hierarchically structured object, such as an automobile, changing some of the parts (for example replacing the brake linings) is normally viewed as a side effect, rather than resulting in a new automobile which has many of the same parts as the original.

The question of side effects is tied up with the phenomenon of naming.[1] As observers of the system, we choose to use the same name for the dining room table and the automobile at two different points in time, despite the fact that they have been modified. The notion of mutable objects thus involves

1. Sussman and Steele give a very good illustration of this point in the context of programming language interpreters in [66].

two aspects: *identity* and *behavior.* The identity of a mutable object is unchangeable; its behavior can change over time.

Syntax

The language we will use to express these ideas formally is called a *situational calculus.* Syntactically, this will be a standard first order logical language with constants, variables, function and relation symbols, logical connectives ($\wedge$, $\vee$, $\neg$, $\supset$ and $\leftrightarrow$), quantifiers ($\forall$ and $\exists$), and equality ($=$ and $\neq$). Set theory ($\in$ and $\notin$) and integer arithmetic (Plus, Times, Gt, etc.) are taken for granted.

Basic Semantic Domains

The identity of a mutable object is embodied in its *name.* The set of names is called **P**. If p is a name, we will commonly say "the object p", rather than more precisely "the object named by p". Names are similar to what are called pointers in computer science.

The universe of possible behaviors is called U. Think of U as a universe of mathematical entities which are used to describe the properties of objects at given points in time. For example, suppose we want to talk about mutable sets; U would then be the universe of mathematical sets.[1] A nice feature of this approach is that U can be treated strictly as formal domain (with an equality relation), i.e. the formal treatment of mutability is independent of the theory of each kind of behavior.

Time is represented as a set of situations, S. Situations are denoted in the language by constant symbols such as s and t. In this section, we are interested only in a notion of time for distinguishing different behaviors of mutable objects. In a later section, a primitive ordering relation on situations will be introduced for specifying the flow of control in computations.

Behavior Functions

The behavior of an object at a given point in time is expressed by a *behavior function*, which maps a name and a situation to a behavior.

$$\mathrm{B}: \mathrm{P} \times \mathrm{S} \rightarrow \mathrm{U}$$

The term B(p,s), where B is a behavior function, may be thought of as expressing the "state of object p at s".[2] The notion of whether or not an object p exists at time s is represented by mapping the behavior of p in some situations to a distinguished element of U called Undefined.

Generally speaking, a computing system provides the user with a set of primitive names and one or more primitive behavior functions, out of which all other mutable objects are built. For example, in Lisp the primitive names are the pointers (addresses) of CONS cells, arrays, and atoms. The primitive behavior functions specify the dotted pair behavior (i.e. the CAR and CDR) of CONS cells at given points in time; the

1. It will later turn out that U and P need not be disjoint, but this assumption makes this initial exposition simpler to understand.
2. We will see later that many different behavior functions can be used, corresponding to different views of an object.

array behavior (i.e. the current function from indices to objects) of array pointers; and the property list of atom pointers.

Equality

Equality in **P**, **S** and **U** is denoted by "=", with the usual rule of substitution. We first consider the intuitive meaning of equality in these three domains, and then discuss how the notion of side effect is represented using equality.

Intuitively, p=q, for two names, p and q, means that p and q are different names for the same mutable object. This could arise, for example, if we introduced two anonymous objects named p and q, and then wanted to consider what would happen if they were the same object.

Intuitively, s=t, for two situations, s and t, means that the behavior of all mutable objects is the same in s and t. We express this formally as the *Axiom of Extensionality for Situations*, which has the following form (where B,C,... are behavior functions).

$$\forall st\, [\, \forall p\, [\, B(p,s)=B(p,t) \wedge C(p,s)=C(p,t) \wedge \ldots\,] \supset s=t\,]$$

For a given computing system it is adequate to include only the primitive behavior functions in this axiom. For example, for Lisp, two situations are equal in which all CONS cells have the same CAR and CDR, all arrays have the same items, and all atoms have the same property lists. Later in this chapter, we will extend this axiom to distinguish situations which are temporally distinct, but in which the behavior of all objects is the same.

Equality in **U** is the equality relation for the particular mathematical domain used to represent behavior. For example, if **U** is sets, then normal set equality is used.

Side Effects

We speak of a side effect having occurred when an object behaves differently in two situations. Formally this is when for some behavior function, B, and situations s and t,

$$B(p,s) \neq B(p,t)$$

We say here that p has been *modified*. For example, to describe the side effect in which the integer 3 is removed from the mutable set p which originally contains the integers 1, 2, and 3 we write the following.

$$\text{set}(p,s)=\{1,2,3\}$$
$$\text{set}(p,t)=\{1,2\}$$

To say that the behavior of an object p is the same in two situations, s and t, we write B(p,s)=B(p,t).

Note that this approach to representing side effects differs from that taken by Green, McCarthy and Hayes. In their calculus, an extra situational variable was added to all the function and relation symbols which described time-dependent aspects of objects. So for example, for a mutable set p they would write member(x,p,s) to assert that x is a member of p at time s. At some other time $t \neq s$, it then might be the case that $\neg$member(x,p,t).

This situational notation becomes awkward, however, when one introduces defined relationships between objects. For example, suppose we wish to assert that between situations s and t some elements may have been removed from set p, but none have been added. The appropriate relation to use here is subset. In Green, McCarthy and Hayes' approach we are forced to define subset as follows, adding two situational arguments:

$$\text{subset}(p,q,s,t) \equiv \forall x\,[\,\text{member}(x,p,s) \supset \text{member}(x,q,t)\,].$$

We then would then assert subset(p,p,t,s) to specify the indicated side effect. In contrast, the situational calculus introduced here allows us to preserve the standard algebra of set relations. So for example we could write

$$\text{set(p,t)} \subseteq \text{set(p,s)}$$

to specify the side effect discussed above.

Behavior Types

In practice, we want to use many different mathematical domains, such as pairs, sequences, sets, integers, lists, etc., to specify the behavior of mutable objects. These sub-domains of **U** are called behavior *types.*

The details of how a behavior type is specified are not important for this level of discussion. For now we can think of a type as providing two things: a predicate on elements of U which distinguishes behaviors of that type from other behaviors, and a rule for determining equality between behaviors of that type. For example, for dotted pairs, the type predicate is Dotted-pairp, and the rule for equality is an axiom which says that two dotted pairs are equal if their CAR and CDR are equal, as shown in Table 8-A.

Associated with each behavior type we usually define a behavior function which maps to elements of that type. For example, **Dotted-pair** is the primitive behavior function of Lisp which specifies the dotted pair behavior of a CONS cell at a given point in time. This function thus has the following relationship to the type predicate Dotted-pairp. (In the following local context Greek letters will be used for elements of **U**.)

$$\forall ps\,[\alpha = \text{dotted-pair}(p,s) \supset [\,\text{dotted-pairp}(\alpha) \vee \alpha = \text{undefined}\,]\,]$$

Table 8-A. Axioms for Dotted Pairs.

Axiom of Extensionality

$$\forall xy\,[\,[\,\text{dotted-pairp}(x) \wedge \text{dotted-pairp}(y) \wedge \text{car}(x) = \text{car}(y) \wedge \text{cdr}(x) = \text{cdr}(y)\,] \supset x = y\,]$$

Axiom of Comprehension

$$\forall xy\,[\,[\,x \neq \text{undefined} \wedge y \neq \text{undefined}\,] \supset \exists z\,[\,\text{dotted-pairp}(z) \wedge \text{car}(z) = x \wedge \text{cdr}(z) = y\,]\,]$$

Alternatively, we can (and will) take the approach of considering the behavior function rather than the type predicate as primitive. For example, for dotted pairs, we can define the type predicate in terms of the behavior function as follows.

$$\text{dotted-pairp}(x) \equiv [\ \exists ps\ \text{dotted-pair}(p,s) = x \wedge x \neq \text{undefined}\ ]$$

In general, for type T (formally a behavior function), we can always write

$$[\ \exists ps\ \mathrm{T}(p,s) = x \wedge x \neq \text{undefined}\ ].$$

where we need to assert a type predicate on x. Furthermore this will be abbreviated[1]

$$\text{instance}(\mathrm{T},x).$$

Function Objects

Many plans in the library are parameterized with respect to functions and relations. For example, a directed graph is modelled as a set of nodes and an edge relation. The accumulation loop plan abstracts away from which particular aggregative function (e.g. Plus, Times, Union) is used. We also need to talk about functions as mutable objects. For example, splicing operations are viewed as side effects to the edge relation of a graph.

In order to formalize such plans, we introduce functions as a behavior type in U. The standard technique for doing this is in a first order language is to introduce the function symbol, **Apply** (and **Binapply** for functions of two arguments, etc.), which is axiomatized as shown in Table 8-B. For basic functions, such as Plus, Times, etc. which we want to use both as first order function symbols and as elements of U, we introduce corresponding underlined symbols such as <u>Plus</u>, <u>Times</u>, etc. with axioms such as the following.

Table 8-B. Axioms for Functions.

Axiom of Extensionality

$$\forall fg\ [\ [\ \text{instance}(\text{function},f) \wedge \text{instance}(\text{function},g) \wedge \forall x\ \text{apply}(f,x) = \text{apply}(g,x)\ ] \supset f = g\ ]$$

Axioms of Comprehension

$$\forall xy\ \exists f [\ \text{instance}(\text{function},f) \wedge \text{apply}(f,x) = y\ ]$$

$$\begin{aligned} \forall fg\ [\ & \text{instance}(\text{function},f) \wedge \text{instance}(\text{function},g) \\ & \supset \exists h\ [\ \text{instance}(\text{function},h) \\ & \qquad \wedge \forall x\ [\ [\ \text{apply}(f,x) = \text{undefined} \supset \text{apply}(h,x) = \text{apply}(g,x)\ ] \\ & \qquad\qquad \wedge [\ \text{apply}(g,x) = \text{undefined} \supset \text{apply}(h,x) = \text{apply}(f,x)\ ]\ ]\ ]\ ] \end{aligned}$$

---

1. Instance must be formally treated as a syntactic abbreviation in order to keep the language first order.

$$\forall x \; \text{apply}(\underline{\text{oneplus}},x)=\text{oneplus}(x)$$

Furthermore, given this convention, we will usually omit the underlining since the underlined symbols can appear only as terms, which are syntactically distinct from function symbols in a first order language.

Relation objects are modelled as boolean valued functions. For example, the element of **U** which corresponds to the arithmetic binary relation, Gt, is axiomatized as follows.

$$\forall xy \, [\, \text{binapply}(\underline{\text{gt}},x,y)=\text{true} \leftrightarrow \text{gt}(x,y) \,]$$

Sequences are treated as functions on a range of integers (basically that a sequence is a function defined for all integers between 1 and the length of the sequence). This makes it convenient to model vectors in Lisp as mutable sequence objects. For example, to describe a STORE operation in which the first item of a vector p is changed from 3 to 4, we write the following.

$$\text{apply}(\text{sequence}(\text{p},\text{s}),1)=3$$
$$\text{apply}(\text{sequence}(\text{p},\text{t}),1)=4$$

As an example of mutable function objects, consider a view of Lisp in which <u>Car</u> and <u>Cdr</u> are the names of mutable function objects, whose domains are CONS cells. In this view, RPLACA and RPLACD are modelled as modifying the function behavior of <u>Car</u> and <u>Cdr</u>, rather than modifying the dotted pair behavior of a given CONS cell. The relationship between these two views is expressed by the following axioms.

$$\forall ps \; \text{apply}(\text{function}(\underline{\text{car}},s),p)=\text{car}(\text{dotted-pair}(p,s))$$
$$\forall ps \; \text{apply}(\text{function}(\underline{\text{cdr}},s),p)=\text{cdr}(\text{dotted-pair}(p,s))$$

8.3 Multiple Points of View

The ability to view the behavior of an object in several different ways is fundamental to the plan calculus. We also need to represent objects whose behavior at a given time depends on the behaviors of other objects at the same time.

As a simple example, suppose we are using a computing system in which mutable sets are not provided as primitives. If mutable sequences are available (either as primitives or themselves built out of some other mutable objects), we can in effect implement a mutable set by viewing a sequence as the set of its range elements. This point of view is defined formally as follows.

$$\sigma=\text{sequence}{>}\text{set}(p,s) \equiv [\, \text{instance}(\text{set},\sigma) \wedge$$
$$\forall \alpha [\, (\alpha \in \sigma) \leftrightarrow \exists i \, [\, \text{apply}(\text{sequence}(p,s),i)=\alpha \wedge \alpha \neq \text{undefined} \,]\,]\,]$$

Sequence>set is a behavior function for sets. Notice that the form of this definition is to construct a set behavior function using a sequence behavior function, as highlighted below.

$$\sigma=\text{sequence}{>}\text{set}(p,s) \equiv [\, \ldots\text{sequence}(p,s)\ldots \,]$$

We can make other definitions of this form to describe how to implement set behavior in terms of list behavior,

$$\sigma = \text{list>set}(p,s) \equiv [\ \ldots\text{list}(p,s)\ldots\ ]$$

and sequence behavior in terms of list behavior,

$$\theta = \text{list>sequence}(p,s) \equiv [\ \ldots\text{list}(p,s)\ldots\ ]$$

and so on. This way of defining behavior functions in terms of other behavior functions is the key idea in representing the implementation of mutable objects.

Constants

The formal system defined above introduces a slight problem with respect to constants. We would like it to be the case that definitions like that of Sequence>set above express both the implementation relationship between mathematical sequences and mathematical sets and between mutable sequences and mutable sets. This problem is solved by extending the functionality of behavior functions so that their first argument may be either a name or an element of U.

$$\mathbf{B}: (\mathbf{P} \cup \mathbf{U}) \times \mathbf{S} \rightarrow \mathbf{U}$$

For each primitive behavior function, such as Set, Sequence and List, we then define an additional axiom which says in effect that constants are immutable objects which behave like themselves.[1] For example, for Sequence we have the following axiom.

$$\forall \theta\ [\ \text{instance}(\text{sequence},\theta) \supset \forall s\ \text{sequence}(\theta,s) = \theta\ ]$$

Sharing

Related to the notion of mutable objects and multiple points of view is the fact that two objects can share structure. The significance of sharing is that side effects on an object propagate to become side effects on other objects with which it shares structure.[2] For example, in Lisp, a single RPLACA can modify the behavior of several different list objects.

Sharing arises out of implementations which involve names. In order to describe such implementations, we need to make names part of U.

$$\mathbf{P} \subset \mathbf{U}$$

In other words we can have pairs of names, sets of names, sequences of names, etc. Given the convention introduced above that behaviors name themselves, this means that the functionality of behavior functions is now simply

---

1. This is similar to the idea in Lisp that constants such as T, NIL and integers, evaluate to themselves.
2. This phenomenon is also sometimes called "aliasing".

$$B: U \times S \rightarrow U.$$

Since we still want to distinguish those elements of U which are not names; we define the set of *constants*, V, as

$$V = U - P.$$

The easiest way to explain how shared structure and the propagation of side effects arises from the use of names is by an example. Consider implementing a (mutable) set as a (mutable) sequence of disjoint (mutable) sets, such that an object is a member of the implemented set iff it is a member of one of the sets in the sequence. This is part of the idea of hash tables, in which the sets in the sequence are called "buckets". This implementation can be defined formally as follows. (In the following local context, Greek letters will now be used to denote constants.)

$$\theta = \text{sequence-of-sets}(p,s) \equiv [\, \text{instance}(\text{sequence},\theta) \wedge$$
$$\forall ij\, [\, i \neq j \supset \text{disjoint}(\text{set}(\text{apply}(\text{sequence}(p,s),i),s),\text{set}(\text{apply}(\text{sequence}(p,s),j),s))\,]\,]$$

$$\sigma = \text{sequence-of-sets>set}(p,s) \equiv [\, \text{instance}(\text{set},\sigma) \wedge$$
$$\forall x\, [\, (x \in \sigma) \leftrightarrow \exists i\, (x \in \text{set}(\text{apply}(\text{sequence}(p,s),i),s))\,]\,]$$

Notice, as highlighted below, that the Set behavior function is used to obtain the set behavior of terms in the sequence.

$$\sigma = \text{sequence-of-sets>set}(p,s) \equiv [\, \ldots\text{set}(\text{apply}(\text{sequence}(p,s),\ldots),s)\ldots\,]$$

This means that the terms in the sequence may be names. By always using behavior functions this way, we provide for the mutability of objects.

Now let us see how this implementation leads to sharing. In particular, let us see how a side effect to any bucket amounts to a side effect to the implemented set. Consider a sequence named H which is viewed as implementing a set according to the technique of Sequence-of-sets>set. Furthermore, suppose B is some bucket of H at some particular time s,

$$\text{apply}(\text{sequence}(H,s),i) = B$$

and that the sequence H is not modified between s and t.

$$\text{sequence}(H,s) = \text{sequence}(H,t)$$

However, if B (and only B) is modified between s and t, i.e.

$$\text{set}(B,s) \neq \text{set}(B,t)$$

$$\forall j\, [\, j \neq i \supset \text{set}(\text{apply}(\text{sequence}(H,s),j),s) = \text{set}(\text{apply}(\text{sequence}(H,s),j),t)\,]$$

it follows from the definitions above that the Sequence-of-sets>set behavior of H is also modified.

$$\text{sequence-of-sets>set}(H,s) \neq \text{sequence-of-sets>set}(H,t)$$

The general point illustrated by this example is that the potential for structure sharing and the propagation of side effects is introduced whenever you start to manipulate names (pointers) as behaviors. It is usual to think of sharing at the lowest level of implementation, such as at the machine language level, or at the CONS cell level in Lisp. This example demonstrates that it may enter in at any level of abstraction.

Sharing does not always arise when pointers are used. For example, suppose we simultaneously view the sequence H above as implementing a set another way, e.g. according to Sequence>set. In this view, for the same situations s and t, no side effect has occurred.

$$\text{sequence>set}(H,s)=\text{sequence>set}(H,t),$$

This is because Sequence>set(H,s) is the set of bucket *names*, which doesn't change even though one of the buckets has been modified. We could give separate names to these two set views of H, as follows.

$$\forall s\ \text{set}(M,s)=\text{sequence-of-sets>set}(H,s)$$
$$\forall s\ \text{set}(K,s)=\text{sequence>set}(H,s)$$

The set M can be thought of as the set of members of the hash table, and K the set of buckets.

Shared List Structure in Lisp

As a second example of sharing, we show how to represent a kind of sharing which should be very familiar to Lisp programmers — shared list structure. This example is more complicated than the hash table example mostly because of the recursive nature of the definition of list behavior. The axioms for lists are given in Table 8-C.

Lists in Lisp are built out of dotted pairs whose Cdr is either Nil (a distinguished constant) or the name of (pointer to) another such dotted pair. This is often called the "linked list" implementation. It is defined formally in terms of behavior functions as follows.

Table 8-C. Axioms for Lists.

Axiom of Extensionality

$$\forall \alpha\beta pqs\,[\,[\alpha=\text{list}(p,s)\wedge\beta=\text{list}(q,s)\wedge\text{head}(\alpha)=\text{head}(\beta)\wedge\text{list}(\text{tail}(\alpha),s)=\text{list}(\text{tail}(\beta),s)\,]\supset\alpha=\beta\,]$$

Axiom of Comprehension

$$\forall xys\,[\,[\,x\neq\text{undefined}\wedge[\,y=\text{nil}\vee\text{list}(y,s)\neq\text{undefined}\,]\,]\leftrightarrow\exists\alpha p\,[\alpha=\text{list}(p,s)\wedge\alpha\neq\text{undefined}\wedge\text{head}(\alpha)=x\wedge\text{tail}(\alpha)=y\,]\,]$$

$$\begin{aligned}\lambda = \text{dotted-pair>list}(p,s) \equiv [\,&[\lambda = \text{nil} \wedge p = \text{nil}\,] \vee \\ &[\,\text{instance}(\text{list},\lambda) \\ &\wedge \text{head}(\lambda) = \text{car}(\text{dotted-pair}(p,s)) \\ &\wedge \text{tail}(\lambda) = \text{dotted-pair>list}(\text{cdr}(\text{dotted-pair}(p,s)),s)\,]\,]\end{aligned}$$

Notice that this is a recursive definition. The tail of the implemented list is the list implemented by the CDR of the dotted pair (in the same way).

To demonstrate how this implementation of lists in Lisp entails structure sharing we show an example of how side effects are propagated. Consider three CONS cells C, D, and E (CONS cells are names with dotted pair behaviors), such that in situation s the Cdr of both C and D is E.

$$\begin{aligned}&\text{cdr}(\text{dotted-pair}(C,s)) = E \\ &\text{cdr}(\text{dotted-pair}(D,s)) = E\end{aligned}$$

If we view C, D and E as implementing lists according to Dotted-pair>list, then by the definitions above, C and D share tails in s, i.e.

$$\text{tail}(\text{dotted-pair>list}(C,s)) = \text{tail}(\text{dotted-pair>list}(D,s))$$

If we now modify the Car of E (e.g. by RPLACA), without changing C and D, so that

$$\begin{aligned}&\text{car}(\text{dotted-pair}(E,s)) \neq \text{car}(\text{dotted-pair}(E,t)) \\ &\text{dotted-pair}(C,s) = \text{dotted-pair}(C,t) \\ &\text{dotted-pair}(D,s) = \text{dotted-pair}(D,t)\end{aligned}$$

it follows that

$$\text{dotted-pair>list}(E,s) \neq \text{dotted-pair>list}(E,t)\,.$$

Furthermore, since they share structure with E viewed as a list, it follows that the list behaviors of C and D have both been modified, i.e.

$$\begin{aligned}&\text{dotted-pair>list}(C,s) \neq \text{dotted-pair>list}(C,t) \\ &\text{dotted-pair>list}(D,s) \neq \text{dotted-pair>list}(D,t)\,.\end{aligned}$$

8.4 Data Plans

We are now in a position to explain the meaning of data plans in terms of the formal framework developed in the preceding sections. The basic idea is that a data plan defines a new type and an associated behavior function. We will first present an example, and then outline the general rules for how to translate from the data plan formalism to a set of axioms in the situational calculus.

Consider the data plan, Segment, shown in Fig. 8-1, which consists of a sequence (the Base) and two natural numbers (Upper and Lower), with the constraint that Upper and Lower are valid indices for the Base, and Lower is less than or equal to Upper.

In terms of the formal framework developed in the preceding sections, the meaning of this plan is to define a new behavior type with the two axioms shown in Table 8-D. The first axiom says that two segments are equal iff their base sequences and upper and lower indices are the same. The second axiom

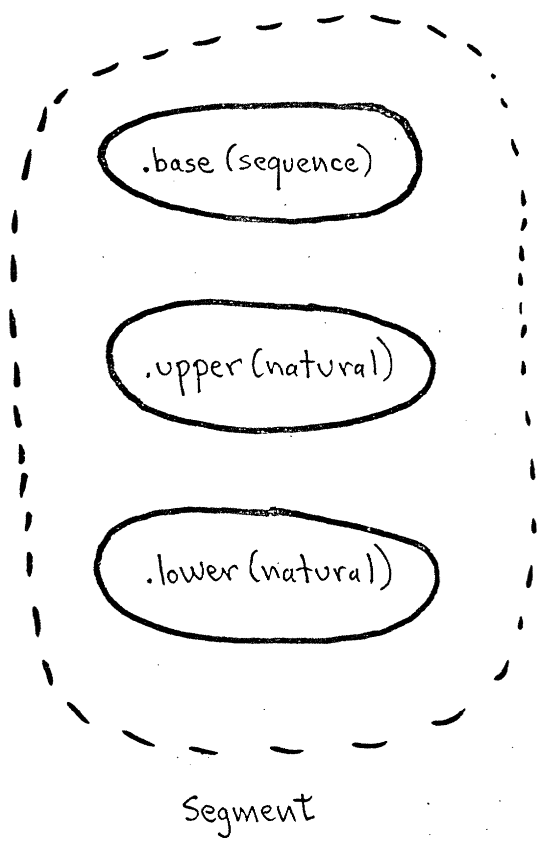


Figure 8-1. Segment Data Plan.

Table 8-D. Segment Data Plan.

Axiom of Extensionality

$$
\begin{aligned}
\forall \alpha\beta pqs\,[\,[\alpha = \text{segment}(p,s) \wedge \beta = \text{segment}(q,s) \\
&\wedge \text{sequence}(\text{base}(\alpha),s) = \text{sequence}(\text{base}(\beta),s) \\
&\wedge \text{natural}(\text{lower}(\alpha),s) = \text{natural}(\text{lower}(\beta),s) \\
&\wedge \text{natural}(\text{upper}(\alpha),s) = \text{natural}(\text{upper}(\beta),s)\,] \\
&\qquad \supset \alpha = \beta\,]
\end{aligned}
$$

Axiom of Comprehension

$$
\begin{aligned}
\forall xyzs\,[\,[\,\text{sequence}(x,s) \neq \text{undefined} \wedge \text{natural}(y,s) \neq \text{undefined} \wedge \text{natural}(z,s) \neq \text{undefined} \\
&\wedge \text{le}(\text{natural}(y,s),\text{natural}(z,s)) \\
&\wedge \text{le}(\text{natural}(y,s),\text{length}(\text{sequence}(x,s))) \\
&\wedge \text{le}(\text{natural}(z,s),\text{length}(\text{sequence}(x,s)))\,] \\
&\qquad \leftrightarrow \exists \alpha p\,[\alpha = \text{segment}(p,s) \wedge \alpha \neq \text{undefined} \\
&\qquad\qquad \wedge \text{base}(\alpha) = x \wedge \text{lower}(\alpha) = y \wedge \text{upper}(\alpha) = z\,]\,]
\end{aligned}
$$

DataPlan Segment
roles .base(sequence) .lower(natural) .upper(natural)
constraints le(.lower,.upper)
$\wedge$ le(.lower,length(.base)) $\wedge$ le(.upper,length(.base))

---

says that for any sequence and two numbers which are valid indices for that sequence, there exists a segment with that sequence as the base and the two numbers as the upper and lower indices; and conversely, that the upper and lower indices of any segment are valid indices for the base sequence and the lower index is less than the upper.

Notice that behavior functions are used throughout these axioms to refer to the behavior of the parts of a segment. This is necessary to allow for shared structure at any level. For example this means that the Base of a segment can be either a sequence of the name of a sequence.

The general rule for translating a data plan into a set of axioms in the situational calculus has two steps. First, the name of the plan formally becomes a behavior function, and the roles of the plan become functions on behaviors of that type. Second, two axioms are written involving these functions.

The first axiom defines equality on the new behavior type in terms of equality of the appropriate behaviors of the roles. So for data plan D with n roles f,g,..., restricted to behavior types T,U,..., respectively, the following axiom schema (called the axiom of extensionality) is written.

$$
\begin{aligned}
\forall \alpha\beta pqs\,[\,[\alpha = \text{D}(p,s) \wedge \beta = \text{D}(q,s) \\
&\wedge \text{T}(\text{f}(\alpha),s) = \text{T}(\text{f}(\beta),s) \wedge \text{U}(\text{g}(\alpha),s) = \text{U}(\text{g}(\beta),s) \wedge \ldots\,] \\
&\qquad \supset \alpha = \beta\,]
\end{aligned}
$$

The second axiom involves the type restrictions on roles of a data plan and the constraints between roles. Formally the constraints are an n-ary relation, where each argument position corresponds to a role, with an extra role for the situation argument to the behavior functions for each role type. So for the same data plan D as above, with constraint relation, C, the following axiom schema (called the axiom of comprehension) is written.

$$\forall sxy... [\,[\, T(x,s)\neq \text{undefined} \wedge U(y,s)\neq \text{undefined} \wedge ... \wedge C(s,x,y,...)\,]$$
$$\leftrightarrow \exists \alpha p\, [\alpha = D(p,s) \wedge \alpha \neq \text{undefined} \wedge f(\alpha)=x \wedge g(\alpha)=y \wedge ...\,]\,]$$

This axiom specifies that instances of the plan D exist, and that all instances satisfy the role type restrictions and constraints.

Finally, the information in the axioms for a data plan can be written in more compact tabular form as shown at the bottom of Table 8-D. This is the notation that will be used in the remainder of this document for formal plan definitions. In this notation, the definition of the constraint relation is made easier to read by using the role names preceded with a leading point (such as ".base") instead of quantified variables corresponding to roles, as appear in the fully written out axioms. Remaining points in constraint formulae are interpreted as normal function application. For example, a path name like ".f.r.s", where f is a role in the plan being defined, is formally equivalent to "s(r(.f))", since r and s are other role functions.

An additional abbreviation used in writing constraints in data plans is to make the behavior functions applied to role functions implicit when the behavior function is the same as the type restriction on the role. For example, at the bottom of Table 8-D,

le(.upper,length(.base))

is an abbreviation for

le(natural(.upper,s),length(sequence(.base,s))) .

The type restriction on each role of a data plan is indicated in the compact notation in parentheses following each role name. For example, the axioms for lists are rewritten using this notation as follows.

DataPlan List
  roles .head(object) .tail(list+nil)

The type List+nil is defined by the behavior function shown below.

$$\lambda = \text{list+nil}(p,s) \equiv [\lambda = \text{list}(p,s) \vee \lambda = \text{nil}\,]$$

The absence of type restriction (other than being defined) is indicated by the keyword "object" after the role name. For example, the axioms for dotted pairs can be rewritten using this notation as follows.

DataPlan Dotted-pair
  roles .car(object) .cdr(object)

8.5 Data Overlays

Intuitively, a data overlay is a many-to-one mapping from one behavior type to another. Formally, a data overlay is a behavior function which is defined in terms of another behavior function. For example, Sequence>set is a data overlay for viewing a sequence as the implementation of a set. Furthermore, because of the way overlays are used in analysis and synthesis, the mapping must be *total* in both directions. For example, for the Sequence>set overlay this means that given any sequence, there exists a set which it implements in this way; and conversely, given any set, there is at least one sequence which implements it in this way. These properties are written formally as the two totality axioms shown in Table 8-E. The definition of Sequence>set is also repeated in this table for reference.

As in the case of data plans, it is more convenient to use a compact tabular notation than to write out the definition and axioms for a data overlay as in Table 8-E. The tabular notation that will be used in the rest of this document for data overlays is shown at the bottom of the table. The general rules for recovering the fully written out formal logical definition and axioms are as follow. In general, the definition of an overlay V from behavior type T to behavior type U,

$$\textbf{\textit{DataOverlay}}\ \mathrm{V} : \mathrm{T} \rightarrow \mathrm{U}$$

is of the following form.

$$y = \mathrm{V}(p,s) \equiv [\ \mathrm{instance}(\mathrm{U},y) \wedge \ldots y \ldots \mathrm{T}(p,s) \ldots\ ]$$

The content of this definition is in the formula relating y and $\mathrm{T}(p,s)$ above. The standard prefix, Instance(U,y), is omitted in the tabular notation. Furthermore, two totality axioms are written from the type information in the header of the tabular notation. These axioms have the following form.

$$\forall xs\ [\ \mathrm{T}(x,s) \neq \mathrm{undefined} \supset \exists y\ [\ y = \mathrm{V}(x,s)\ ]\ ]$$
$$\forall ys\ [\ \mathrm{U}(y,s) \neq \mathrm{undefined} \supset \exists x\ [\ y = \mathrm{V}(x,s)\ ]\ ]$$

Table 8-E. Sequence as Set Overlay.

Totality Axioms

$$\forall xs\ [\ \mathrm{sequence}(x,s) \neq \mathrm{undefined} \supset \exists y\ [\ y = \mathrm{sequence{>}set}(x,s)\ ]\ ]$$

$$\forall ys\ [\ \mathrm{set}(y,s) \neq \mathrm{undefined} \supset \exists x\ [\ y = \mathrm{sequence{>}set}(x,s)\ ]\ ]$$

Definition

$$y = \mathrm{sequence{>}set}(p,s) \equiv [\ \mathrm{instance}(\mathrm{set},y) \wedge \forall \alpha\ [\ (\alpha \in y) \leftrightarrow \exists i\ \mathrm{apply}(\mathrm{sequence}(p,s),i) = \alpha\ ]\ ]$$

DataOverlay Sequence>set: sequence → set
definition $y = \mathrm{sequence{>}set}(p,s) \equiv \forall \alpha[\ (\alpha \in y) \leftrightarrow \exists i\ \mathrm{apply}(\mathrm{sequence}(p,s),i) = \alpha\ ]$

8.6 Computations

In the plan calculus, computations are thought of as structures, some of whose parts are elements of S (situations) and some of whose parts are elements of U (mutable objects and constants). In order to formally describe computations in the situational calculus, we introduce a new domain, **C**, of computations. **C** is divided into types which are specified by axioms similar to those used to specify behavior types in U. In the rest of this section, after some formal preliminaries, we present axioms for various computation types. In the next section we use these foundations to specify the semantics of the temporal plan formalism.

Temporal Order

Thus far we have been using situations only as arguments to behavior functions to distinguish the different states of objects. In order to represent temporal order in computations we introduce a new primitive relation, called Precedes, which is formally a total order on S. Intuitively, this relation captures the notion of states occurring "before" or "after" other states. This relation also makes it possible to talk about cyclic computations in which all objects return to the same state as at some earlier time. Formally, this is achieved by extending the Axiom of Extensionality for Situations as follows.

$$\begin{aligned} &\forall st\,[\,[\,\forall p\,[\,B(p,s)=B(p,t) \wedge C(p,s)=C(p,t) \wedge \ldots\,] \\ &\qquad \wedge\, \forall u\,[\,\mathrm{precedes}(s,u) \leftrightarrow \mathrm{precedes}(t,u)\,]\,]\supset s=t\,] \end{aligned}$$

B,C,... here are the appropriate primitive behavior functions as before. This axiom says that two situations are identical iff the behavior of all objects is the same and they are indistinguishable in the temporal order.

Note that Precedes is a *total* order. This is because we are formally dealing with sequential computations. As we will see shortly, however, in specifying computation *types* we will often leave the order between two steps unconstrained.

Termination

Another basic feature of computations we need to deal with is termination. In order to talk about this formally, we introduce a bottom element in S, i.e.

$$\forall s\ \mathrm{precedes}(s,\perp)\,.$$

Intuitively, $\perp$ represents a computation step which is never reached. As we shall see in the following sections, $\perp$ appears in the axioms for elementary computation types, such as operations and tests. The termination properties of composite types, such as loops, are then derived from the axioms of the components and their connections. Important termination properties are whether or not a given step is reached in all instances of a computation type, and whether there exists an instance of a computation type in which a given step is reached. Formally these properties amount to the claim that the situations in question are not equal to $\perp$.

Operations

The most basic computation types are operations. Operations in general involve two situations, one of which precedes the other, and some number of input and output objects. An example of such a type is set addition operations. Intuitively, a set addition computation is an operation involving three objects: the old set, the new set, and the member added. This is specified formally by the two axioms shown in Table 8-F. These axioms involve the type predicate, Set-add, and the functions In, Out, Old, New, and Input, on elements of the type which act like the role functions of a data structure (e.g. Head and Tail for lists). For example, consider two situations, s and t, and mutable sets A and B, such that the following statements hold.

precedes(s,t)
set(A,s) = {1,2}
set(B,t) = {1,2,3}

Formally, what we have here is a computation, α, such that

set-add(α)
in(α) = s
out(α) = t
old(α) = A
input(α) = 3
new(α) = B.

Table 8-F. Set Addition Operations.

Axiom of Extensionality

$$\forall\alpha\beta\,[\,[\,\text{set-add}(\alpha) \wedge \text{set-add}(\beta) \wedge \text{in}(\alpha)=\text{in}(\beta) \wedge \text{out}(\alpha)=\text{out}(\beta) \\ \wedge\ \text{old}(\alpha)=\text{old}(\beta) \wedge \text{input}(\alpha)=\text{input}(\beta) \wedge \text{new}(\alpha)=\text{new}(\beta)\,] \supset \alpha=\beta\,]$$

Axiom of Comprehension

$$\forall xyzst\,[\,[\,\text{precedes}(s,t) \wedge [\,s\neq\bot \supset \\ [\,t\neq\bot \wedge \text{set}(x,s)\neq\text{undefined} \wedge \text{set}(y,t)\neq\text{undefined} \wedge z\neq\text{undefined} \\ \wedge\ (z \in \text{set}(y,t)) \\ \wedge\ \forall w\,[\,w\neq z \supset [\,(w \in \text{set}(y,t)) \leftrightarrow (w \in \text{set}(x,s))\,]\,]\,]\,]\,] \\ \leftrightarrow \exists\alpha\,[\,\text{set-add}(\alpha) \wedge \text{in}(\alpha)=s \wedge \text{out}(\alpha)=t \\ \wedge\ \text{old}(\alpha)=x \wedge \text{new}(\alpha)=y \wedge \text{input}(\alpha)=z\,]\,]$$

IOspec Set-add / .old(set) .input(object) $\Rightarrow$ .new(set)
postconditions (.input $\in$ .new)
$\wedge\ \forall x\,[\,x\neq$.input $\supset [\,($.input $\in$ .new$) \leftrightarrow ($.input $\in$ .old$)\,]\,]$

In the following local context Greek letters will be used to denote elements of **C**. Note that we will also informally refer to elements of **C** as *instances* of a computation type, T. Formally, this just means T(α).

The first axiom in Table 8-F defines equality of set addition operations in terms of equality of the situations and objects involved. The second axiom specifies a necessary and sufficient condition between the objects and situations of set addition operations. These axioms amount to what is standardly called an *input-output specification.*

Let us now pay attention to the details of the second axiom in Table 8-F. Part of the necessary and sufficient condition deals with the temporal order and termination properties of set addition operations, as shown below. (This pattern of specification is followed for operations in general).

$$[\,\mathrm{precedes}(s,t) \wedge [\, s \neq \perp \supset [\, t \neq \perp \wedge \ldots]]]$$

Thus the In situation precedes the Out situation. Furthermore, if the In situation is reached, it follows that the Out situation is reached, i.e. the operation always terminates. Notice that it follows from this axiom and the definition of $\perp$ that, if the In situation is never reached (i.e. $s=\perp$), then the Out situation is never reached ($t=\perp$).

The remainder of the condition part of the second axiom specifies that the members of the New set in the Out situation are exactly the members of the Old set in the In situation, with the sole addition of the Input object. This relationship is conditionalized inside $t \neq \perp$ to avoid contradiction in the case when neither situation is reached, i.e. $s=t=\perp$.

Notice that this specification uses the Set behavior function in referring to the Old and New objects. This means that instances of this computation type include both operations in which the input and output sets are distinct objects, and those which involve a side effect (e.g. suppose old(α)=new(α) in the example above). More will be said about plans involving side effects at the end of this chapter.

A more compact tabular notation for writing input-output specifications is shown at the bottom of Table 8-F. The first line of this notation lists the name of the operation type (formally a predicate on computations), separated by a slash from the input roles, separated by a double arrow from the output roles. Type restrictions are indicated in parentheses following the role names, as in the compact data plan notation. Roles are formally functions on computations. To recover the formal axioms from this notation for a general input-output specification, P, with input roles f,g,..., and output roles m,n,..., we first write an axiom of extensionality of the following form.

$$\begin{aligned} \forall \alpha\beta\, [\, [\, & \mathrm{P}(\alpha) \wedge \mathrm{P}(\beta) \wedge \mathrm{in}(\alpha)=\mathrm{in}(\beta) \wedge \mathrm{out}(\alpha)=\mathrm{out}(\beta) \\ & \wedge \mathrm{f}(\alpha)=\mathrm{f}(\beta) \wedge \mathrm{g}(\alpha)=\mathrm{g}(\beta) \wedge \ldots \wedge \mathrm{m}(\alpha)=\mathrm{m}(\beta) \wedge \mathrm{n}(\alpha)=\mathrm{n}(\beta) \wedge \ldots] \\ & \supset \alpha=\beta\,] \end{aligned}$$

The constraint between roles in an input-output specification is made easier to read in the compact tabular notation by using the role names preceded with a leading point instead of quantified variables, similar to the constraint notation for data plans. Embedded points are interpreted as normal functional nesting.

Like the compact notation for data plans, the application of behavior functions corresponding to role types is also made implicit in compact input-output specifications. For example, at the bottom of Table 8-F

(.input ∈ .new)

is an abbreviation for

(.input ∈ set(.new,.in)).

By convention, the situational argument to such implicit behavior function applications is either ".in" or ".out", depending on whether the role involved is an input or an output. No behavior function is supplied for roles, such as Input, without type restriction (indicated by the keyword Object as in data plans).

After expanding all abbreviations as outlined above, the constraint relation is formally a relation, C, where each argument position corresponds to a role, plus two situational arguments which correspond to In and Out. In general for an input-output specification P with input roles f,g,..., with type restrictions T,U,..., and output roles m,n,..., with type restrictions A,B,...,we then write the following axiom.

$$
\begin{aligned}
\forall stxy...vw...\ [\,[\,&\text{precedes}(s,t) \wedge [\, s \neq \perp \supset \\
&[\, t \neq \perp \wedge \mathrm{T}(x,s) \neq \text{undefined} \wedge \mathrm{U}(y,s) \neq \text{undefined} \wedge ... \\
&\wedge \mathrm{A}(v,t) \neq \text{undefined} \wedge \mathrm{B}(w,t) \neq \text{undefined} \wedge ... \\
&\wedge \mathrm{C}(s,t,x,y,...,v,w,...)\,]\,]\,] \\
\leftrightarrow \exists \alpha\ [\, &\mathrm{P}(\alpha) \wedge \text{in}(\alpha) = s \wedge \text{out}(\alpha) = t \\
&\wedge \mathrm{f}(\alpha) = x \wedge \mathrm{g}(\alpha) = y \wedge ... \wedge \mathrm{m}(\alpha) = v \wedge \mathrm{n}(\alpha) = w \wedge ...\,]\,]
\end{aligned}
$$

Finally, note that the constraint clauses in an input-output specification are divided into those which involve only input roles (called preconditions), and those which involve both input and output roles (called postconditions). For example, the following is the compact specification of **@Function,** the operation of applying a function to an argument to get the corresponding range element.

IOspec @Function / .op(function) .input(object) ⇒ .output(object)
preconditions apply(.op,.input)≠undefined
postconditions apply(.op,.input)=.output

Tests

The second basic computation type in the plan calculus is tests. Tests in general have three situational roles: an input situation, In, and two alternative following situations, Succeed and Fail, only one of which is reached in any instance.

An example of a type of test, membership tests, is shown specified formally in Table 8-G. The first axiom is the usual axiom of extensionality which defines equality on a computation type in terms of equality of its roles. The roles of a membership test are the three situational roles, In, Succeed and Fail, and two object roles, Universe (a set) and Input.

Table 8-G. Membership Tests.

Axiom of Extensionality

$$\forall \alpha\beta\ [\ [\ \text{member?}(\alpha) \wedge \text{member?}(\beta) \wedge \text{in}(\alpha)=\text{in}(\beta) \wedge \text{succeed}(\alpha)=\text{succeed}(\beta) \wedge \text{fail}(\alpha)=\text{fail}(\beta)$$
$$\wedge\ \text{universe}(\alpha)=\text{universe}(\beta) \wedge \text{input}(\alpha)=\text{input}(\beta)\ ] \supset \alpha=\beta\ ]$$

Axiom of Comprehension

$$\begin{aligned}
\forall xystu\ [\ [\ &\text{precedes}(s,t) \wedge \text{precedes}(s,u) \wedge [\ t=\perp \vee u=\perp\ ]\\
&\wedge [\ s\neq\perp \supset [\ [\ t\neq\perp \vee u\neq\perp\ ]\\
&\qquad \wedge\ x\neq\text{undefined} \wedge \text{set}(y,s)\neq\text{undefined}\\
&\qquad \wedge [\ t\neq\perp \leftrightarrow (x \in \text{set}(y,s))\ ]\ ]\ ]\ ]\\
\leftrightarrow \exists\alpha\ [\ &\text{member?}(\alpha) \wedge \text{in}(\alpha)=s \wedge \text{succeed}(\alpha)=t \wedge \text{fail}(\alpha)=u\\
&\wedge \text{universe}(\alpha)=y \wedge \text{input}(\alpha)=x\ ]\ ]
\end{aligned}$$

Test Member? / .universe(set) .input(object)
condition (.input ∈ .universe)

The second axiom in Table 8-G says roughly that membership tests succeed if the Input is a member of the Universe; otherwise they fail. This is expressed formally by specifying the conditions under which the Succeed and Fail roles are equal to $\perp$, as shown below.

$$\begin{aligned}
[\ &\text{precedes}(s,t) \wedge \text{precedes}(s,u) \wedge [\ t=\perp \vee u=\perp\ ]\\
&\wedge [\ s\neq\perp \supset [\ [\ t\neq\perp \vee u\neq\perp\ ] \wedge \ldots \wedge [\ t\neq\perp \leftrightarrow \ldots\ ]\ ]\ ]\ ]
\end{aligned}$$

This is the pattern of specification used in general for tests. At most one of either Succeed or Fail is reached in any instance. If the condition of the test is true in the In situation, then the Succeed situation is reached; if it is false, then the Fail situation is reached. If the In situation is never reached, it follows that neither Succeed nor Fail are reached.

In the next section, we will see how tests specified this way can be combined with other computations, via the notion of control flow, to construct specifications for larger conditional computations.

Finally, Table 8-G shows an example of the compact notation for tests. The header line lists the name of the computation type followed by the object role names with type restrictions, similar to the input-output specification notation introduced in the preceding section. The axiom of extensionality which follows from this notation in general is obvious. The axiom of comprehension for a test P? with object roles f,g,..., and type restrictions T,U,..., is of the following form.

$$
\begin{aligned}
\forall stuxy... [\, [\, &\text{precedes}(s,t) \wedge \text{precedes}(s,u) \wedge [\, t=\perp \vee u=\perp \,] \\
&\wedge [\, s\neq\perp \supset [\, [\, t\neq\perp \vee u\neq\perp \,] \\
&\qquad \wedge \mathrm{T}(x,s)\neq\text{undefined} \wedge \mathrm{U}(y,s)\neq\text{undefined} \wedge ... \\
&\qquad \wedge [\, t\neq\perp \leftrightarrow \mathrm{C}(s,x,y,...) \,]\,]\,]\,] \\
\leftrightarrow \exists\alpha\, [\, &\mathrm{P?}(\alpha) \wedge \text{in}(\alpha)=s \wedge \text{succeed}(\alpha)=t \wedge \text{fail}(\alpha)=u \\
&\wedge \text{f}(\alpha)=x \wedge \text{g}(\alpha)=y \wedge ... \,]\,]
\end{aligned}
$$

The relation C above is derived by expanding abbreviations in the ***condition*** part of the compact test notation in the same way abbreviations are expanded in the preconditions and postconditions of an input-output specification, supplying ".in" as the situational argument to implicit behavior functions where required.

8.7 Temporal Plans

In this section we extend **C** by allowing parts of computations to be not only situations and objects, but also other computations. This gives us the ability to combine already defined computation types, such as operations and tests, into the specification of larger computations. For example, we can define a computation type which has two steps. The first step is an instance of @Discrimination;[1] the second step is a membership test (Member?). The temporal plan representation of this computation type is shown in Fig. 8-2. The axioms which are the formal translation of this plan are given in Table 8-H.

Table 8-H. Discriminate and Member Plan.

Axiom of Extensionality

$$
\begin{aligned}
\forall\alpha\beta\, [\, [\, &\text{discriminate+member?}(\alpha) \wedge \text{discriminate+member?}(\beta) \\
&\wedge \text{discriminate}(\alpha)=\text{discriminate}(\beta) \wedge \text{if}(\alpha)=\text{if}(\beta) \,] \supset \alpha=\beta \,]
\end{aligned}
$$

Axiom of Comprehension

$$
\begin{aligned}
\forall\alpha\beta\, [\, [\, &\text{@discrimination}(\alpha) \wedge \text{member?}(\beta) \wedge \text{cflow}(\text{out}(\alpha),\text{in}(\beta)) \\
&\wedge \text{set}(\text{output}(\alpha),\text{out}(\alpha))=\text{set}(\text{universe}(\beta),\text{in}(\beta)) \\
&\wedge \text{input}(\alpha)=\text{input}(\beta) \,] \\
&\qquad \leftrightarrow \exists\delta\, [\, \text{discriminate+member?}(\delta) \wedge \text{discriminate}(\delta)=\alpha \wedge \text{if}(\delta)=\beta \,]\,]
\end{aligned}
$$

TemporalPlan discriminate+member?
roles .discriminate(@discrimination) .if(member?)
constraints cflow(.discriminate.out,.if.in)
$\wedge$ .discriminate.output=.if.universe $\wedge$ .discriminate.input=.if.input

---

1. @Discrimination is a specialization of @Function in which the function applied (Op) is a discrimination, and therefore the Output is a set.

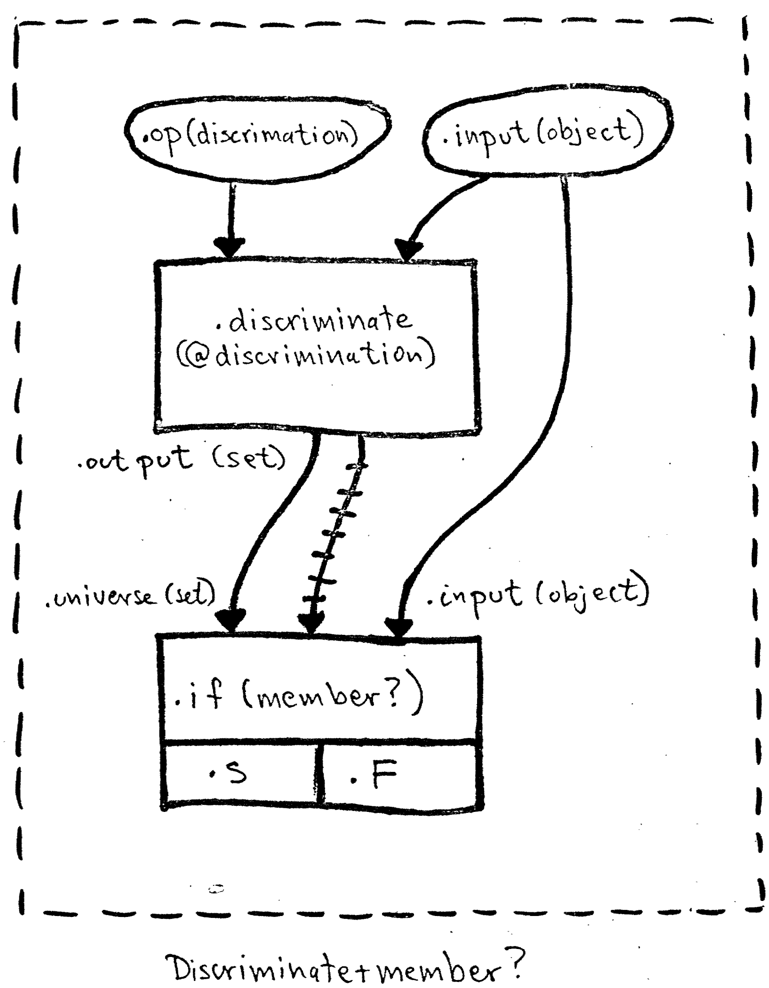


Figure 8-2. A Temporal Plan

Notice that the name of the plan, **Discriminate+member?**, is formally a predicate on computations. The roles of the plan, Discriminate and If, are formally functions on computations, like Old, In, Input, New, etc. in the preceding section. The ranges of these role functions, however, are computations, as can be seen in the second axiom of Table 8-H highlighted below.

$$\begin{array}{l} \forall \alpha\beta\ [\ [\text{@discrimination}(\alpha) \wedge \text{member?}(\beta) \ldots\ ] \\ \qquad \leftrightarrow\ \exists\delta\ [\ \text{discriminate+member?}(\delta) \wedge \text{discriminate}(\delta) = \alpha \wedge \text{if}(\delta) = \beta\ ]\ ] \end{array}$$

Table 8-I. Bump and Update Plan.

Axiom of Extensionality

$$\begin{array}{l} \forall \alpha\beta\ [\ [\ \text{bump+update}(\alpha) \wedge \text{bump+update}(\beta) \wedge \text{bump}(\alpha) = \text{bump}(\beta) \wedge \text{update}(\alpha) = \text{update}(\beta) \\ \qquad \wedge\ \text{old}(\alpha) = \text{old}(\beta) \wedge \text{new}(\alpha) = \text{new}(\beta)\ ] \supset \alpha = \beta\ ] \end{array}$$

Axiom of Comprehension

$$\begin{array}{l} \forall xy\alpha\beta\ [\ [\text{@oneminus}(\alpha) \wedge \text{newterm}(\beta) \\ \qquad \wedge\ \text{upper-segment}(x,\text{in}(\alpha)) \neq \text{undefined} \\ \qquad \wedge\ \text{upper-segment}(y,\text{out}(\beta)) \neq \text{undefined} \\ \qquad \wedge\ \text{cflow}(\text{out}(\alpha),\text{in}(\beta)) \\ \qquad \wedge\ \text{upper-segment}(x,\text{in}(\alpha)) = \text{upper-segment}(x,\text{in}(\beta)) \\ \qquad \wedge\ \text{upper-segment}(y,\text{out}(\alpha)) = \text{upper-segment}(y,\text{out}(\beta)) \\ \qquad \wedge\ \text{integer}(\text{input}(\alpha),\text{in}(\alpha)) = \text{natural}(\text{lower}(\text{upper-segment}(x,\text{in}(\alpha))),\text{in}(\alpha)) \\ \qquad \wedge\ \text{sequence}(\text{old}(\beta),\text{in}(\beta)) = \text{sequence}(\text{base}(\text{upper-segment}(x,\text{in}(\beta))),\text{in}(\beta)) \\ \qquad \wedge\ \text{integer}(\text{output}(\alpha),\text{out}(\alpha)) = \text{natural}(\text{arg}(\beta),\text{in}(\beta)) \\ \qquad \wedge\ \text{integer}(\text{output}(\alpha),\text{out}(\alpha)) = \text{natural}(\text{lower}(\text{upper-segment}(y,\text{out}(\alpha))),\text{out}(\alpha)) \\ \qquad \wedge\ \text{sequence}(\text{new}(\beta),\text{out}(\beta)) = \text{sequence}(\text{base}(\text{upper-segment}(y,\text{out}(\beta))),\text{out}(\beta))\ ] \\ \qquad\qquad \leftrightarrow\ \exists\delta\ [\ \text{bump+update}(\delta) \\ \qquad\qquad\qquad \wedge\ \text{bump}(\delta) = \alpha \wedge \text{update}(\delta) = \beta \wedge \text{old}(\delta) = x \wedge \text{new}(\delta) = y\ ]\ ] \end{array}$$

TemporalPlan Bump+update
roles .bump(@oneminus) .update(newterm) .old(upper-segment) .new(upper-segment)
constraints cflow(.bump.out,.update.in)
$\wedge$ $.\text{old}_{.\text{bump.in}} = .\text{old}_{.\text{update.in}}$
$\wedge$ $.\text{new}_{.\text{bump.out}} = .\text{old}_{.\text{update.out}}$
$\wedge$ .bump.input=.old.lower
$\wedge$ .update.old=.old.base
$\wedge$ .bump.output=.update.arg
$\wedge$ .bump.output=.new.lower
$\wedge$ .update.new=.new.base

In general, the type restriction on a role in a temporal plan is either a behavior type (formally a behavior function) or a computation type (formally a predicate on computations). An example of a temporal plan which has some of both kinds of roles is shown in Fig. 8-3. The Old and New roles are restricted to be instances of the Upper-segment data plan;[1] Bump and Update are operations.[2] The axioms for this plan are shown in Table 8-I. The axiom of comprehension in this table is quite long, but is of the same general form as the axiom of comprehension for Discriminate+member?. The first three lines stipulate type restrictions. For temporal roles, these are assertions of the appropriate computation type predicates, e.g.

$$\text{@oneminus}(\alpha) \wedge \text{newterm}(\beta)\,.$$

For behavior type roles, the assertion of a type restriction has to include the situation in which it is used, e.g.

$$\text{upper-segment}(x,\text{in}(\alpha)) \neq \text{undefined}$$
$$\text{upper-segment}(y,\text{out}(\beta)) \neq \text{undefined}\,.$$

For data roles that are used in more than one place, additional equalities are added to guarantee that the data object is the same in all situations of use. For example, the two lines following the control flow constraint in the comprehension axiom of Bump+update are for this purpose.

$$\text{upper-segment}(x,\text{in}(\alpha)) = \text{upper-segment}(x,\text{in}(\beta))$$
$$\text{upper-segment}(y,\text{out}(\alpha)) = \text{upper-segment}(y,\text{out}(\beta))$$

The remaining equalities have to do with data flow, which will be discussed later in this section.

Control Flow

Control flow constraints (hatched arrows in plan diagrams) are formalized in the situational calculus as follows.

$$\text{cflow}(s,t) \equiv [\,\text{precedes}(s,t) \wedge [\,s=\perp \leftrightarrow t=\perp\,]\,]$$

In other words, control flow implies temporal order and termination is preserved. However, the two situations do not have to be equal.

Each control flow arc in a temporal plan becomes a Cflow clause in the axiom of comprehension for the computation type. The terms in this clause are the appropriate In, Out, Succeed or Fail roles, as read from the diagram. For example, the control flow arc in Fig. 8-2 becomes the following clause in the comprehension axiom of Table 8-H.

$$\text{cflow}(\text{out}(\alpha),\text{in}(\beta))$$

---

1. Upper-segment is a specialization of Segment in which the Upper index is equal to the length of the sequence.
2. The specifications for @Oneminus and Newterm can be found in the appendix.

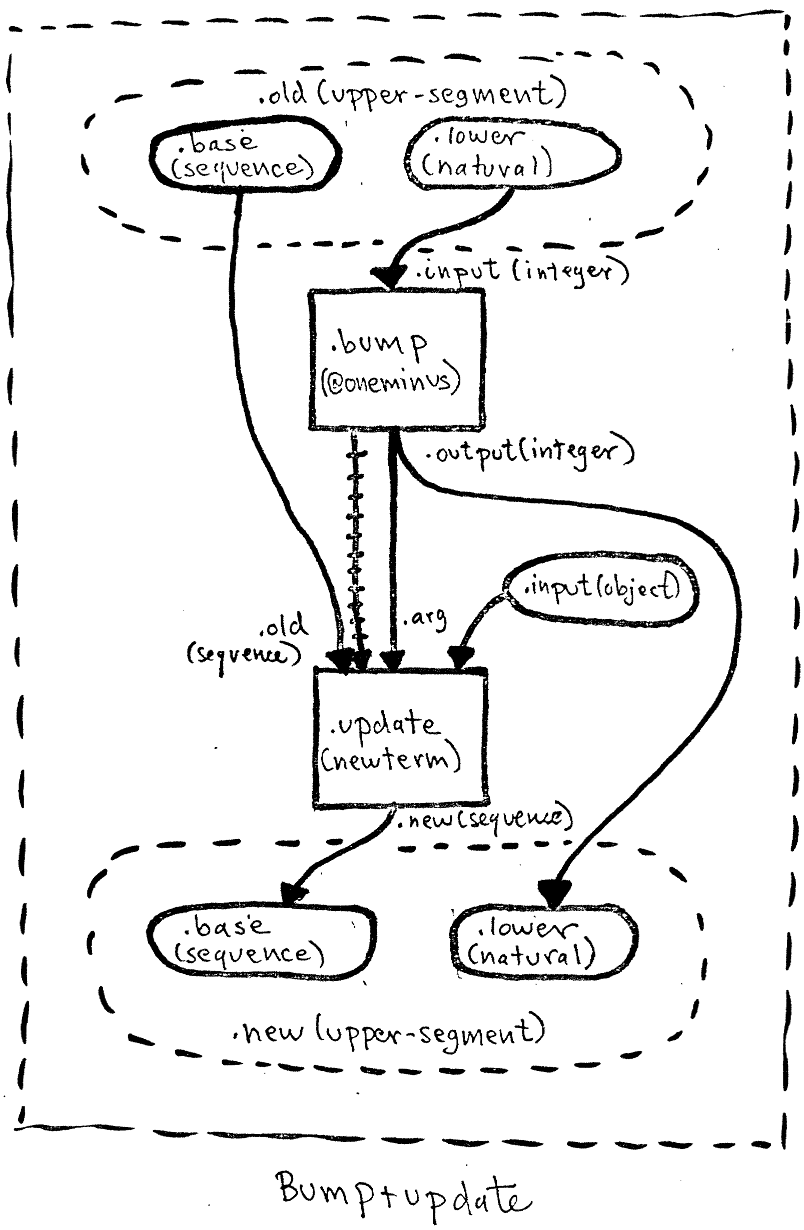


Figure 8-3. Another Temporal Plan.

Data Flow

A second kind of "glue" in temporal plans is data flow. Data flow arcs in general are translated into equalities between names and behaviors in different situations. The details of this translation, however, depend on whether the data flow is between operations and tests, or whether it also involves data plan roles, such as Old and New in Bump+update.

We start with the simple case of the data flow arc in Discriminate+member? (Fig. 8-2) from Discriminate.Input to If.Input. This arc is translated into the following clause in the comprehension axiom for this plan.

$$\text{input}(\alpha)=\text{input}(\beta)$$

This is an example of data flow between the untyped roles of two operations. In other words, what is being passed between these two operations is being treated as a name. The other data flow arc in Fig. 8-2 is between Discriminate.Output (a set) and If.Universe (a set). For typed roles, the rules is to write the equality in terms of the behavior function and the appropriate situational role, such as In or Out, e.g.

$$\text{set}(\text{output}(\alpha),\text{out}(\alpha))=\text{set}(\text{universe}(\beta),\text{in}(\beta))\,.$$

The distinction between whether or not a data flow equality involves a behavior function is similar to the distinction between "call by reference" and "call by value" in some programming languages.

Fig. 8-3 shows data flow involving data plan roles. In particular, different parts of Old and New are inputs and outputs of Bump and Update. These data flows are translated into the equalities listed on separate lines of the comprehension axiom in Table 8-1. The first of these is

$$\text{integer}(\text{input}(\alpha),\text{in}(\alpha))=\text{natural}(\text{lower}(\text{upper-segment}(x,\text{in}(\alpha))),\text{in}(\alpha))\,.$$

This is the translation of the arc from Old.Lower to Bump.Input in the plan representation. Notice how the behavior functions have been supplied on both sides above,[1] and that the situational arguments are the In situation of the consuming operation.

$$\text{sequence}(\text{old}(\beta),\text{in}(\beta))=\text{sequence}(\text{base}(\text{upper-segment}(x,\text{in}(\beta))),\text{in}(\beta))$$

The next data flow arc, shown above, is from Old.Base to Update.Old. Here again we have behavior functions on both sides, with the same situational argument, namely Update.In. The translation of the two data flow arcs involving New are similar, as shown below.

$$\text{integer}(\text{output}(\alpha),\text{out}(\alpha))=\text{natural}(\text{lower}(\text{upper-segment}(y,\text{out}(\alpha))),\text{out}(\alpha))$$
$$\text{sequence}(\text{new}(\beta),\text{out}(\beta))=\text{sequence}(\text{base}(\text{upper-segment}(y,\text{out}(\beta))),\text{out}(\beta))$$

1. The input and output of @Oneminus are of type integer. Natural is a specialization of Integer.

Finally, examples of the compact notation for writing the axioms for temporal plans are shown at the bottom of Table 8-H and Table 8-I. In general for a temporal plan P with roles f,g,..., we write the following axiom of extensionality.

$$\forall\alpha\beta\ [\,[\,P(\alpha) \wedge P(\beta) \wedge f(\alpha)=f(\beta) \wedge g(\alpha)=g(\beta) \wedge \ldots\,] \supset \alpha=\beta\,]$$

The axiom of comprehension is of the following form, where f,g,..., are temporal roles with types T,U,... and m,n,... are data roles with types A,B,... .

$$\begin{aligned}&\forall xy\ldots vw\ldots\ [\,[\,T(x) \wedge U(y) \wedge \ldots\\&\quad \wedge A(v,\ldots)\neq\text{undefined} \wedge B(w,\ldots)\neq\text{undefined} \wedge \ldots\\&\quad \wedge C(x,y,\ldots,v,w,\ldots)\,]\\&\qquad\qquad \leftrightarrow \exists\alpha\ [\,P(\alpha) \wedge f(\alpha)=x \wedge g(\alpha)=y \wedge \ldots \wedge m(\alpha)=v \wedge m(\alpha)=w \wedge \ldots\,]\,]\end{aligned}$$

The constraint relation C above is derived by expanding abbreviations in the constraints of the compact notation in a similar manner to the way abbreviations are expanded in compact input-output specifications. In particular, implicit applications of behavior functions with appropriate situational arguments are provided for path names which terminate in roles typed by behavior functions. For example,

.if.universe

in the constraints of Discriminate+member? is expanded to

set(.if.universe,.if.in) .

C also includes constraints that guarantee an object used in more than one situation is the same in all situations of use. Table 8-I illustrates all these conventions. To facilitate comparison, the constraints of Bump+update in the compact notation are written in the same order line by line as in the fully written out axiom above in the table. The first two lines of the compact notation in Table 8-I following the control flow constraint illustrate how situational arguments can be explicitly indicated in the compact notation by subscripts.

Conditional Plans

Fig. 8-4 is an example of a conditional plan which computes absolute value.[1] The formal axioms for this plan, written in compact notation, are as follow.

1. Lt-zero? is the test for less than zero. @Negative computes the negative of an integer.

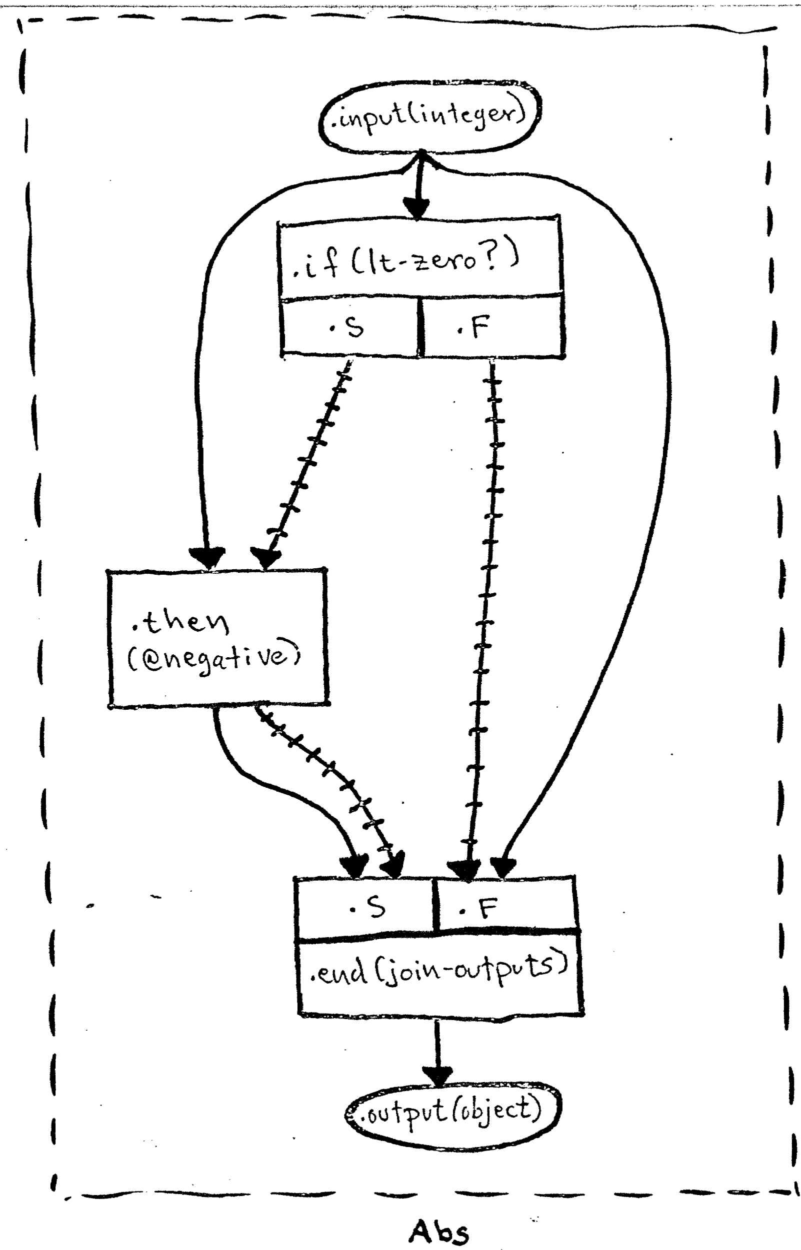


Figure 8-4. A Conditional Plan.

TemporalPlan Abs
  roles .if(lt-zero?) .then(@negative) .end(join-output)
  constraints cflow(.if.succeed,.then.in)
    $\wedge$ cflow(.then.out,.end.succeed)
    $\wedge$ cflow(.if.fail,.end.fail)
    $\wedge$ .if.input = .then.input
    $\wedge$ .then.output = .end.succeed-input
    $\wedge$ .if.input = .end.fail-input

This plan has two key features which are typical of conditional plans in general. First, notice the control flow arc from If.Succeed to Then.In. The intuitive meaning of this arc is that the @Negative operation is to be performed only if the test succeeds. This is expressed formally as the following property of the Abs plan, which follows from the way tests, operations and control flow have been axiomatized.

$$\forall\alpha\ [\ \mathrm{abs}(\alpha) \supset [\ \mathrm{in}(\mathrm{then}(\alpha)) \neq \perp\ \leftrightarrow\ \mathrm{lt}(\mathrm{input}(\mathrm{if}(\alpha)),0)\ ]\ ]$$

Second, notice the data flow and control flow arcs involving the join (End). The meaning of these arcs is that the output of the join is either If.Input or Then.Output, depending on whether the test succeeds or fails. Stated formally, we want the Abs plan to have the following property.

$$\begin{aligned}&\forall\alpha\ [\ \mathrm{abs}(\alpha) \supset \\ &\quad [\ [\ \mathrm{lt}(\mathrm{input}(\mathrm{if}(\alpha)),0) \supset \mathrm{output}(\mathrm{end}(\alpha)) = \mathrm{negative}(\mathrm{input}(\mathrm{if}(\alpha)))\ ] \\ &\quad \wedge [\neg\mathrm{lt}(\mathrm{input}(\mathrm{if}(\alpha)),0) \supset \mathrm{output}(\mathrm{end}(\alpha)) = \mathrm{input}(\mathrm{if}(\alpha))\ ]\ ]\ ]\end{aligned}$$

This is achieved by axiomatizing joins (with one output) as shown in Table 8-J. Joins are like the mirror images of tests. Joins have three situational roles, Succeed, Fail, and Out. Like tests, at most one of either Succeed or Fail is reached in any instance. Unlike tests, however, joins do not represent any real computation, since the Out situation is always equal to either the Succeed or Fail situation, depending on which is reached. The purpose of the join is to state, in a modular fashion, the connection between which

---

Table 8-J. Joining Outputs.

Axiom of Extensionality

$$\begin{aligned}&\forall\alpha\beta\ [\ [\ \mathrm{join\text{-}output}(\alpha) \wedge \mathrm{join\text{-}output}(\beta) \\ &\quad \wedge \mathrm{succeed}(\alpha) = \mathrm{succeed}(\beta) \wedge \mathrm{fail}(\alpha) = \mathrm{fail}(\beta) \wedge \mathrm{out}(\alpha) = \mathrm{out}(\beta) \\ &\quad \wedge \mathrm{succeed\text{-}input}(\alpha) = \mathrm{succeed\text{-}input}(\beta) \wedge \mathrm{fail\text{-}input}(\alpha) = \mathrm{fail\text{-}input}(\beta)\ ] \\ &\qquad \supset \alpha = \beta\ ]\end{aligned}$$

Axiom of Comprehension

$$\begin{aligned}&\forall stuxyz\ [\ [\ [\ t = \perp \vee u = \perp\ ] \wedge [\ t = \perp \supset [\ s = u \wedge x = z\ ]\ ] \wedge [\ u = \perp \supset [\ s = t \wedge x = y\ ]\ ]\ ] \\ &\quad \leftrightarrow \exists\alpha\ [\ \mathrm{join\text{-}output}(\alpha)\ \wedge \mathrm{out}(\alpha) = s \wedge \mathrm{succeed}(\alpha) = t \wedge \mathrm{fail}(\alpha) = u \\ &\qquad \wedge \mathrm{output}(\alpha) = x \wedge \mathrm{succeed\text{-}input}(\alpha) = y \wedge \mathrm{fail\text{-}input}(\alpha) = z\ ]\ ]\end{aligned}$$

whether a test succeeds or fails and which of two possible outputs is made available for further computation. The two possible outputs are the Succeed-input and Fail-input roles of the join. One of these is equal to the Output role (which one depends on whether Succeed or Fail is reached), from which data flow arcs to following computations emanate.

8.8 Temporal Overlays

A temporal overlay is formally a function from one computation type to another. Furthermore, like data overlays, this mapping must be total in both directions. For example, consider the temporal overlay shown in Fig. 8-5, which expresses how to view instances of the temporal plan Discriminate+member? as implementing membership tests on a set implemented as a discrimination function.

The formal definition and totality axioms for this overlay are given in Table 8-K. Each correspondence in the figure becomes an equality in the formal definition. Unlabelled correspondences, such as between Discriminate.Input on the left and Input on the right become simple equalities such as

$$\text{input}(\beta) = \text{input}(\text{discriminate}(\alpha))\,.$$

Table 8-K. Implementing Membership in a Discrimination

Totality Axioms

$\forall\alpha$ [discriminate+member?(α) $\supset$ $\exists\beta$ [member?(β) $\wedge$ β = discriminate+member?>member?(α)]]

$\forall\beta$ [member?(β) $\supset$ $\exists\alpha$ [discriminate+member?(β) $\wedge$ β = discriminate+member?>member?(α)]]

Definition

β = discriminate+member?>member?(α) $\equiv$ [member?(β)
$\wedge$ set(universe(β),in(β)) = discrimination>set(op(discriminate(α)),in(discriminate(α)))
$\wedge$ input(β) = input(discriminate(α))
$\wedge$ in(β) = in(discriminate(α))
$\wedge$ fail(β) = fail(if(α))
$\wedge$ succeed(β) = succeed(if(α))]

TemporalOverlay Discriminate+member?>member?: discriminate+member? → member?
correspondences
member?.universe = discrimination>set(discriminate+member?.discriminate.op)
$\wedge$ member?.input = discriminate+member?.discriminate.input
$\wedge$ member?.in = discriminate+member?.discriminate.in
$\wedge$ member?.fail = discriminate+member?.if.fail
$\wedge$ member?.succeed = discriminate+member?.if.succeed

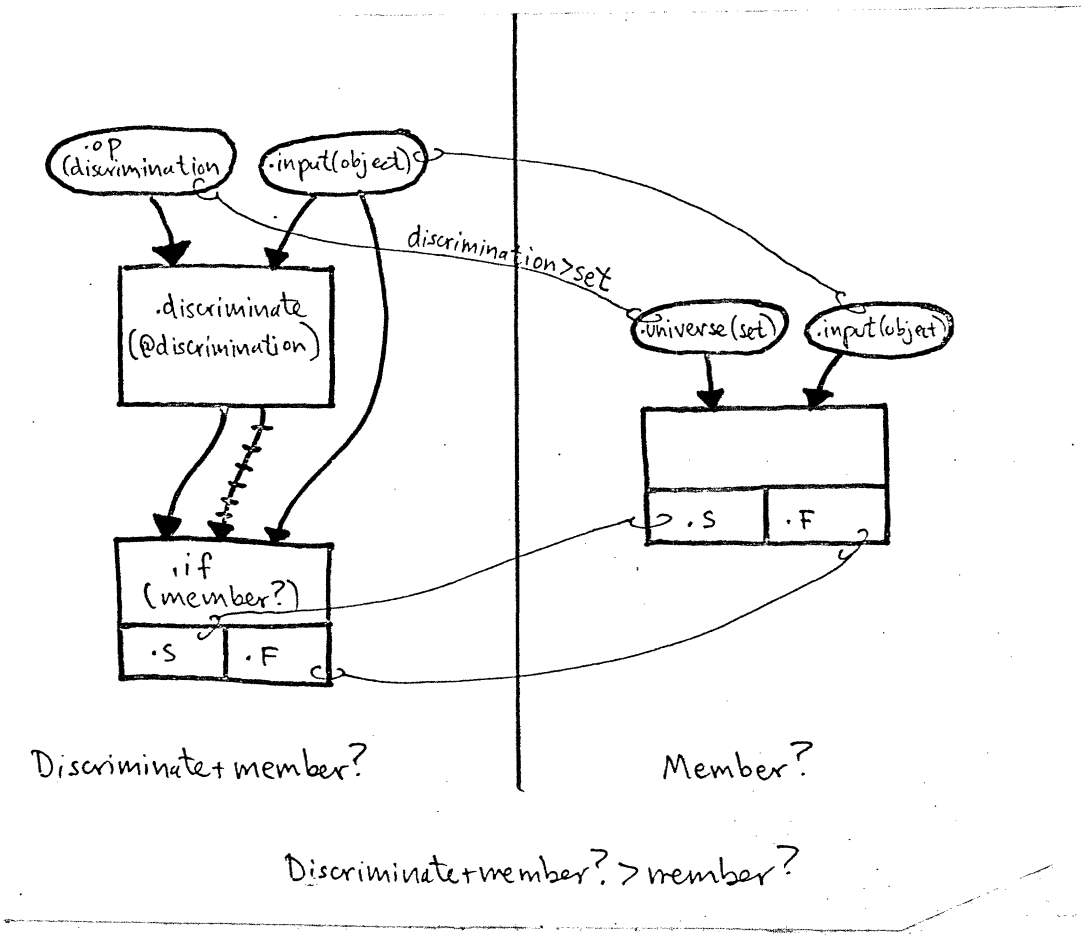


Figure 8-5. A Temporal Overlay.

Since overlays can be used in defining other overlays, some correspondences in temporal overlays are labelled with the names of other overlays. For example, the correspondence between Discriminate.Op on the left and Universe on the right is labelled with the Discrimination>set overlay.[1] Intuitively, this means that Discriminate+member?.Discriminate.Op is viewed as implementing Member?.Universe according to Discrimination>set. This is written formally in the definition of Discriminate+member?>member? as follows.

$$\text{set}(\text{universe}(\beta),\text{in}(\beta)) = \text{discrimination>set}(\text{op}(\text{discriminate}(\alpha)),\text{in}(\text{discriminate}(\alpha)))$$

Notice that behavior functions are supplied for typed roles with the appropriate situational arguments as usual. In general, the definition of an overlay V from computation type T to computation type U, where f,g,... are the role functions of U, is of the following form.

$$\beta = \text{V}(\alpha) \equiv [\ \text{U}(\beta) \wedge \text{f}(\beta) = ...\alpha... \wedge \text{g}(\beta) = ...\alpha... \wedge ...\ ]$$

In other words, there is an equality for each role of β in terms of some function of α. This form, together with the extensionality axiom of U, guarantees the uniqueness property of the function V.

As with data overlays, it is more convenient to use a compact tabular notation than to write out the definition and axioms for a temporal overlay as in Table 8-K. An example of the tabular notation is shown at the bottom of the table. In general, from the header line

TemporalOverlay V: T → U

the following two totality axioms are written.

$$\forall\alpha\ [\ \text{T}(\alpha) \supset \exists\beta\ [\ \text{U}(\beta) \wedge \beta = \text{V}(\alpha)\ ]\ ]$$
$$\forall\beta\ [\ \text{U}(\beta) \supset \exists\alpha\ [\ \text{T}(\alpha) \wedge \beta = \text{V}(\alpha)\ ]\ ]$$

The definition of the overlay function is abbreviated in the tabular notation by listing only the equalities and leaving behavior functions and situational arguments implicit in the usual way.

8.9 Specialization and Extension

In this section we discuss two additional ways of making use of already defined plans in defining new ones, namely, specialization and extension.

Specialization

The basic idea of specialization is to define a type whose instances are a subset of another type. A common motivation for doing this is to exploit the properties of the subtype in some particular implementation. For example, we have earlier in this chapter defined a general data plan, Segment, involving an upper and lower index to a base sequence. One way of implementing a mutable stack is to use an instance of Segment in which only the lower index is varied — the upper index is always equal to

1. This is a data overlay similar to Sequence-of-sets>set introduced earlier in this chapter.

the length of the base sequence. We called this plan **Upper-segment**. The formal relation between Upper-segment and Segment is captured by the following statement.

$$\sigma = \text{upper-segment}(p,s) \equiv [\sigma = \text{segment}(p,s) \wedge \text{natural}(\text{upper}(\sigma),s) = \text{length}(\text{sequence}(\text{base}(\sigma),s))]$$

Thus Upper-segment is Segment with an additional constraint. In tabular notation, this will be written as follows.

DataPlan Upper-segment ***specialization*** segment
roles .base(sequence) .lower(natural) .upper(natural)
constraints .upper = length(.base)

Notice that a specialization has the same roles as the more general plan, and that the application of behavior functions of the appropriate type for each role is abbreviated in the constraints in usual manner.

The specialization of computation types is similar. For example, the following is the general input-output specifications for finding a node in a directed graph (Digraph), which satisfies a given predicate.

IOspec Digraph-find / .universe(digraph) .criterion(predicate) ⇒ .output(object)
preconditions $\exists x$ [node(.universe,x) ∧ apply(.criterion,x) = true]
postconditions node(.universe,.output) ∧ apply(.criterion,.output) = true

An important special case of directed graphs are threads, in which each node has a unique successor and there are no cycles. Finding nodes in threads is considerably simpler than the general case. The computation type of such operations is specified formally as follows.

$$\text{thread-find}(\alpha) \equiv [\ \text{digraph-find}(\alpha) \wedge \text{thread}(\text{old}(\alpha),\text{in}(\alpha)) \neq \text{undefined}\ ]$$

Thus the additional constraint here is an additional type restriction on the Old role. (The behavior function Thread is the appropriate specialization of Digraph.) This is written in the compact tabular notation as follows.

IOspec Thread-find / .universe(thread) .criterion(predicate) ⇒ .output(object)
specialization digraph-find

Of course, computation types can also be specialized by additional constraints between roles. For example, set addition by side effect, #Set-add, is viewed as a specialization set addition in general. This is expressed formally as follows.

$$\#\text{set-add}(\alpha) \equiv [\ \text{set-add}(\alpha) \wedge \text{old}(\alpha) = \text{new}(\alpha)\ ]$$

In other words, instances of #Set-add are those instances of Set-add in which the Old and New set objects are identical. In tabular notation, this will be written as follows.

IOspec #Set-add / .old(object) .input(object) ⇒ .new(object) ***specialization*** set-add
postconditions .old = .new

Notice that the type restrictions on Old and New above are Object rather than Set, as in Set-add. This usage is essentially a syntactic trick to control the abbreviation that will be applicable in the postcondition above. Logically, an Object restriction is weaker than a Set restriction, so no information is added.

Extension

The basic idea of extension is to define a new type with an additional role function, such that instances of the new type have the same constraints as the old type between those roles which are in common. The formalization of extension is more complicated than the formalization of specialization in the preceding section. In the case of specialization, the new behavior function or predicate on computations can be defined simply in terms of the old one. For extension, however, new extensionality and comprehension axioms need to be written for the new type. However, these new axioms are related to those of the old type in a systematic way.

A common use of extension is to add an additional output to an input-output specification. For example, when Thread-find operations are used in conjunction with other plans, such as splicing, it is convenient to have as output not only the node found, but also the previous node in the thread. We call this extra role Previous, and the extended operation type **Internal-thread-find.**

Table 8-L. Internal Thread Find.

Axiom of Extensionality

$$\begin{aligned}
\forall \alpha\beta\ [\,[\ & \text{internal-thread-find}(\alpha) \wedge \text{internal-thread-find}(\beta) \wedge \text{in}(\alpha)=\text{in}(\beta) \wedge \text{out}(\alpha)=\text{out}(\beta) \\
& \wedge \text{universe}(\alpha)=\text{universe}(\beta) \wedge \text{criterion}(\alpha)=\text{criterion}(\beta) \wedge \text{output}(\alpha)=\text{output}(\beta) \\
& \underline{\wedge\ \text{previous}(\alpha)=\text{previous}(\beta)}\ ] \\
& \qquad \supset \alpha=\beta\ ]
\end{aligned}$$

Axiom of Comprehension

$$\begin{aligned}
\forall wxyzst\ [\,[\ & \exists\alpha\ [\ \text{thread-find}(\alpha) \wedge \text{in}(\alpha)=s \wedge \text{out}(\alpha)=t \\
& \qquad \wedge \text{universe}(\alpha)=w \wedge \text{criterion}(\alpha)=x \wedge \text{output}(\alpha)=y\ ] \\
& \underline{\wedge\ z\neq\text{undefined}} \\
& \underline{\wedge\ \text{apply}(\text{predicate}(x,s),\text{root}(\text{thread}(w,s)))=\text{false}} \\
& \underline{\wedge\ \text{successor}(\text{thread}(w,s),z,y)}\ ] \\
& \qquad \leftrightarrow \exists\beta\ [\ \text{internal-thread-find}(\beta) \wedge \text{in}(\beta)=s \wedge \text{out}(\beta)=t \\
& \qquad\qquad \wedge \text{universe}(\beta)=w \wedge \text{criterion}(\beta)=x \wedge \text{output}(\beta)=y \wedge \text{previous}(\beta)=z\ ]\,]
\end{aligned}$$

IOspec Internal-thread-find / .universe(thread) .criterion(predicate)
$\Rightarrow$ .output(object) .previous(object)
extension thread-find
preconditions apply(.criterion,root(.universe)) = false
postconditions successor(.universe,.previous,.output)

The axioms for Internal-thread-find are shown in Table 8-L. They are derived from the axioms of Thread-find by adding the underlined portions. In the axiom of extensionality, an additional equality is added for the Previous role. The axiom of comprehension specifies the constraints on the new type by first referring to the corresponding instance of the old type,

$$\exists\alpha\,[\,\text{thread-find}(\alpha) \wedge \text{in}(\alpha)=s \wedge \ldots\,]$$

and then specifying the added underlined constraints, which include the type restriction on the new role and some additional constraints between this role and the others. One could think of this as an extension step, followed by a specialization, but in practice one almost never adds a new role without relating it to the old roles. As usual the more compact tabular notation is shown at the bottom of the table.

8.10 Plans Involving Side Effects

Given the logical foundations of the plan calculus described in this chapter, it is now possible to explain a little more about plans involving side effects. This section has two basic points to make. The first point is that, since reasoning about plans involving side effects can in general be quite difficult,[1] the inspection method approach is to formalize many common forms of side effect usage as plans and overlays in the plan library and use them in the analysis, synthesis and verification of programs to bypass general reasoning.

The second basic point is that, whenever possible, plans in the library are written at a level of abstraction which does not make any commitment to whether or not side effects are used. This principle is exemplified by the input-output specifications shown in Table 8-M. In each case, the "impure" specification is viewed as a specialization of the pure specification. For example, #Set-add is the specification for adding a member to a set by side effect.[2]

Table 8-N and Fig. 8-6 show an example of a very general form of side effect usage which can be captured using plans and overlays. The right hand side of the overlay in Table 8-N is the plan for modifying a function (Update.Old) such that all domain elements which used to map to a given range element (Update.Value) now map to a new element (Update.Input), where the new range element is computed from the old range element by the Action. For example, this is the structure of the SYMBOL-TABLE-ADD procedure in Chapter Two: a new bucket is computed from an old bucket of the table by Set-add; the table (modelled as function from indices to buckets) is then modified so that the new bucket is the new value of the index of the old bucket.

The overlay #Old+input+new>action+update records the fact that computing the new range element by side effect obviates the step of modifying the function itself. This is the way the SYMBOL-TABLE-DELETE procedure works (except for the special case handled separately before the loop): the new bucket is computed from the old bucket by side effect (splicing out), so that no subsequent

1. See Shrobe [64] for an approach to the explicit control of reasoning about plans involving side effects.
2. Note that the prefix character "#" is used to name impure input-output specifications as a mnemonic device.

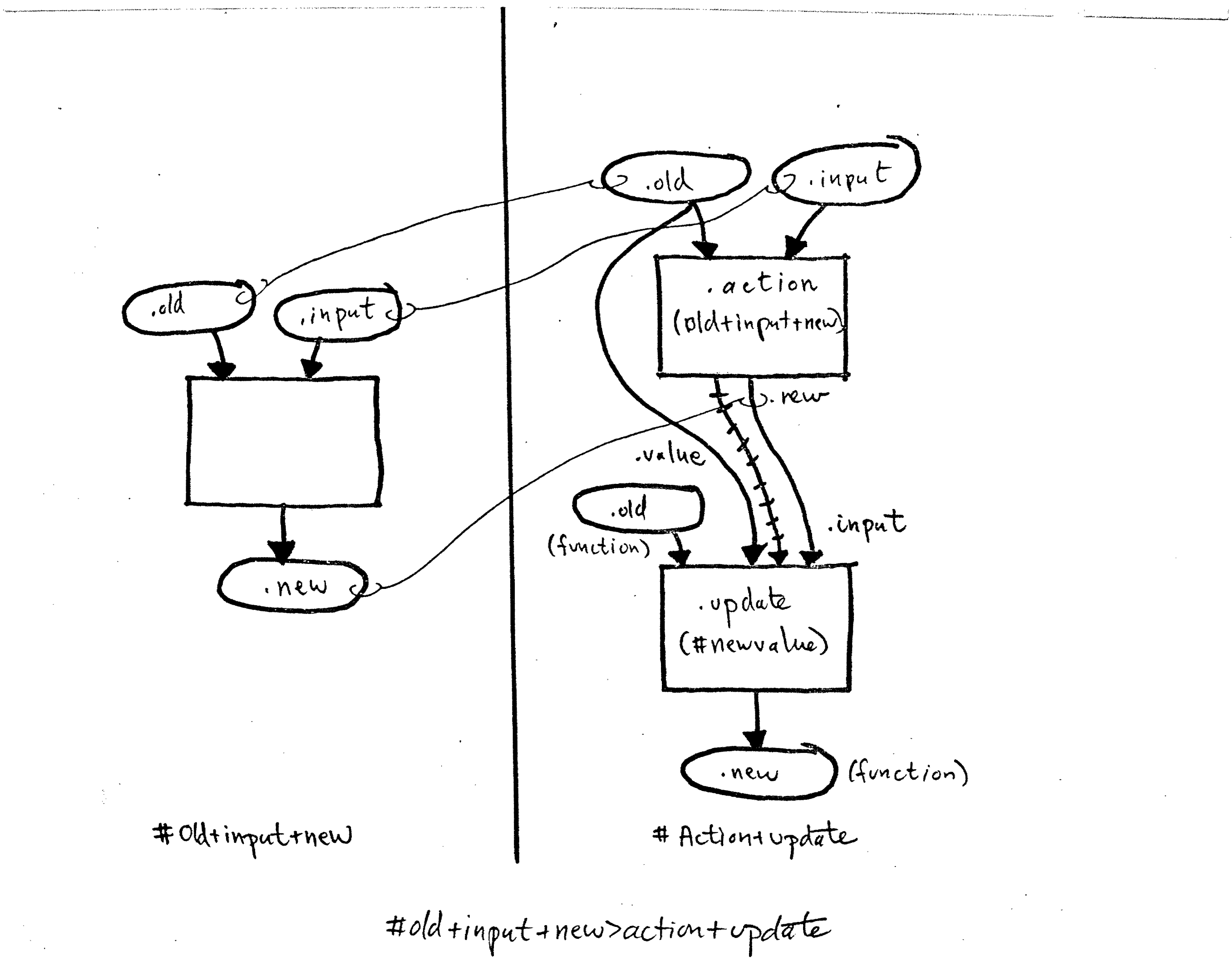


Figure 8-6. Updating a Function by Side Effect.

Table 8-M. Impure Input-Output Specifications.

IOspec **old+new** / .old(object) ⇒ .new(object)

IOspec #**old+new** / .old(object) ⇒ .new(object) *specialization* old+new
postconditions .old = .new

IOspec #**set-add** / .old(set) .input(object) ⇒ .new(set)
specialization set-add #old+new

IOspec #**set-remove** / .old(set) .input(object) ⇒ .new(set)
specialization set-remove #old+new

IOspec #**restrict** / .old(set) .criterion(predicate) ⇒ .new(set)
specialization restrict #old+new

IOspec #**newarg** / .old(function) .arg(object) .input(object) ⇒ .new(function)
specialization newarg old+new

IOspec #**newvalue** / .old(function) .value(object) .input(object) ⇒ .new(function)
specialization newvalue #old+new

ARRAYSTORE is required. Other specializations and extensions of #Old+input+new for which this implementation works are shown as properties of the overlay.

Table 8-N. Updating a Function by Side Effect.

TemporalOverlay #**old+input+new>action+update:** #old+input+new → #action+update
  properties ∀*AP* [*P* = #old+input+new>action+update(*A*) ⊃
      [[instance(#set-add,*A*) ↔ instance(#set-add,*P*.action)]
      ∧ [instance(#set-remove,*A*) ↔ instance(#set-remove,*P*.action)]
      ∧ [instance(#newright,*A*) ↔ instance(#newright,*P*.action)]
      ∧ [instance(#newleft,*A*) ↔ instance(#newleft,*P*.action)]
      ∧ [instance(#internal-thread-add,*A*) ↔ instance(#internal-thread-add,*P*.action)]
      ∧ [instance(#internal-thread-remove,*A*)
          ↔ instance(#internal-thread-remove,*P*.action)]]]
  correspondences #old+input+new.old = #action+update.action.old
      ∧ #old+input+new.input = #action+update.action.input
      ∧ #old+input+new.in = #action+update.action.in
      ∧ #old+input+new.out = #action+update.update.out

TemporalPlan #**action+update**
  roles .action(old+input+new) .update(#newvalue)
  constraints .action.old = .update.value ∧ .action.output = .update.input
      ∧ cflow(.action.out,.update.in)

IOspec **old+input+new** / .old(object) .input(object) ⇒ .new(object) *extension* old+new

IOspec #**old+input+new** / .old(object) .input(object) ⇒ .new(object)
  extension #old+new
  specialization old+input+new

CHAPTER NINE

LOOPS AND TEMPORAL ABSTRACTION

9.1 Introduction

The plan calculus uses self-referential (i.e. recursive) definitions to represent unbounded structures. This chapter concentrates on the special case of singly recursive plans, and loops in particular. The generalization of these ideas to multiple recursion will be discussed briefly at the end of the chapter.

We begin in Table 9-A with a minimal plan, Single-recursion, which says nothing more than that there is a role, Tail, constrained to be either Nil or itself a Single-recursion. A finite single recursion is defined as one whose tail is Nil or eventually has a Nil tail. "Eventually" is defined by the transitive closure tail relation, Tail*, which is in turn defined in terms of the n-th tail relation, Tailn.

The most common singly recursive data plan, List, has already been discussed in Chapter Eight. The next section in this chapter will concentrate on how loops, the most common singly recursive temporal plans, are represented in the plan calculus. The section following then shows how to represent the relationship between singly recursive temporal plans (loops) and and singly recursive data plans (lists) using overlays. Finally, note that the taxonomy of loops discussed in this chapter covers only loops with a single jump from the end of the loop to the beginning (i.e. interleaved loops are not included).

Table 9-A. Single Recursion.

DataPlan **single-recursion**
  roles .tail(single-recursion+nil)

Type **single-recursion+nil** *uniontype* single-recursion nil

DataPlan **finite-single-recursion** *specialization* single-recursion
  roles .tail(single-recursion+nil)
  constraints .tail = nil $\vee$ $\exists x$ [tail*(.tail) = x $\wedge$ x = nil]

Function **tail***: single-recursion $\rightarrow$ object
  properties $\forall R$ [instance(single-recursion,tail*(R)) $\vee$ tail*(R) = nil]
  definition x = tail*(R) $\equiv$ [$\exists n$ tailn(n,R) = x]

Binfunction **tailn**: natural $\times$ single-recursion $\rightarrow$ single-recursion+nil
  definition x = tailn(n,R) $\equiv$ [[n = 1 $\wedge$ x = R.tail] $\vee$ x = tailn(oneminus(n),R.tail)]

9.2 Loops

Since the temporal order relation on situations (Precedes) is not allowed to have any cycles, loops are represented in the plan calculus as singly recursive plans where the jump from the end of the loop to the beginning is viewed as a recursive invocation. For example, Fig. 9-1 is the plan diagram for the SEARCHLIST program below.

```
(DEFINE SEARCHLIST
   (LAMBDA (L P)
      (PROG (ENTRY)
         LP (SETQ ENTRY (CAR L))
            (COND ((FUNCALL P ENTRY)(RETURN ENTRY)))
            (SETQ L (CDR L))
            (GO LP))))
```

The Tail role, which represents the recursive invocation of the loop body, is constrained to be an instance of the same plan as the outside plan. This is indicated in plan diagrams by a spiral line from the outside plan box to the recursive role. The operation boxes in the diagram are instances of @Function; the test boxes are instances of @Predicate; and the join boxes are instances of Join-output. Thus we are viewing the program as if it were coded as follows.

```
(DEFINE SEARCHLIST
   (LAMBDA (L P)
      (PROG (ENTRY)
            (LP))))

(DEFINE LP
   (LAMBDA ()
            (SETQ ENTRY (CAR L))
            (COND ((FUNCALL P ENTRY)(RETURN ENTRY)))
            (SETQ L (CDR L))
            (LP)))
```

This form of single recursion, in which the recursive call is the last step of the program, is often called "tail recursion" or "iterative" single recursion. Many Lisp interpreters and compilers treat loops and tail recursions as superficial syntactic variations. For example, in Scheme, a dialect of Lisp developed by Sussman and Steele [65], the PROG with GO construction is provided as a macro which expands into a single recursion similar to the example above. The Scheme interpreter executes tail recursions without accumulating stack depth. The compiler for Scheme also views looping constructs as macros which expand into singly recursive structures.

Given this view of loops, it is possible to formalize a small set of the basic plans which decompose many loops into intuitively meaningful parts. The remainder of this section will present these plans, along with explanations and some typical fragments of code which are instances. The taxonomy of loops presented here is an extension of the work of Waters [73].

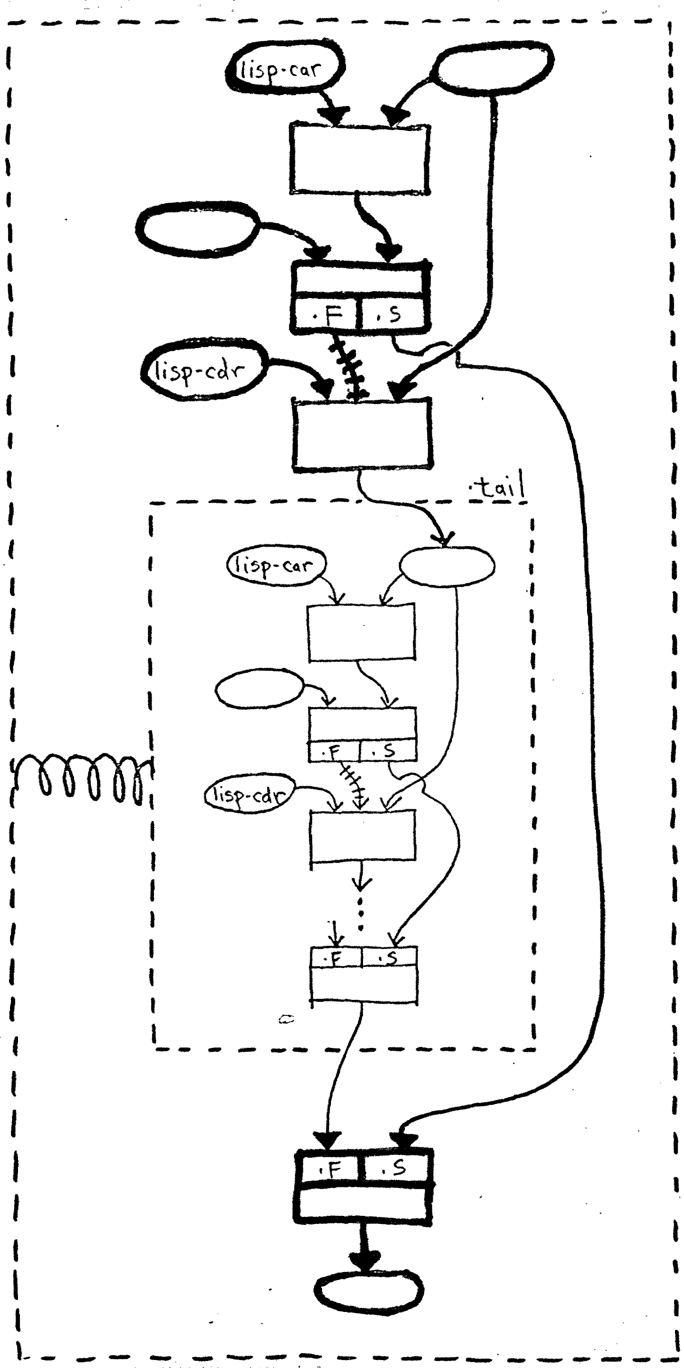


Figure 9-1. A Tail Recursive Temporal Plan.

Table 9-B. Iterative Steady State Plans.

TemporalPlan **iterative-application** *extension* single-recursion
roles .action(@function) .tail(iterative-application)
constraints .action.op = .tail.action.op ∧ cflow(.action.out,.tail.action.in)

TemporalPlan **iterative-generation** *specialization* iterative-application
roles .action(@function) .tail(iterative-generation)
constraints .action.output = .tail.action.input

TemporalPlan **iterative-filtering** *extension* single-recursion
roles .filter(cond) .tail(iterative-filtering)
constraints instance(@predicate,.filter.if)
∧ .filter.if.criterion = .tail.filter.if.criterion
∧ cflow(.filter.end.out,.tail.filter.if.in)

Steady State Plans

To begin let us ignore any exits from a loop and the question of termination. This is what I call the "steady state" model. This viewpoint will be formalized later as an overlay which explicitly assumes that the loop does not exit.

One of the most common computations in a loop is to repeatedly apply a given function (the same function each time) to the output of the preceding application of that function. This pattern of application is in general (i.e. for multiple recursive plans) called *generation.* The special case for loops is called iterative generation, as shown in Table 9-B and Fig. 9-2. Notice that the starting value for the generation is Action.Input, the input to the first application. The SEARCHLIST example contains an instance of Iterative-generation, as shown below, where the function being applied is Cdr and the variable L holds the successive values generated.

```
(PROG (...)
   LP ...
      (SETQ L (CDR L))
      (GO LP))))
```

Using SETQ this way is the most common way of coding iterative generation in Lisp, but there are other ways of achieving the necessary data flow, as illustrated below.

```
(DEFINE LP
   (LAMBDA (L)
      ...
      (LP (CDR L))))
```

A particularly common specialization of iterative generation is Counting, where the function applied is Oneplus and the initial input is 1.

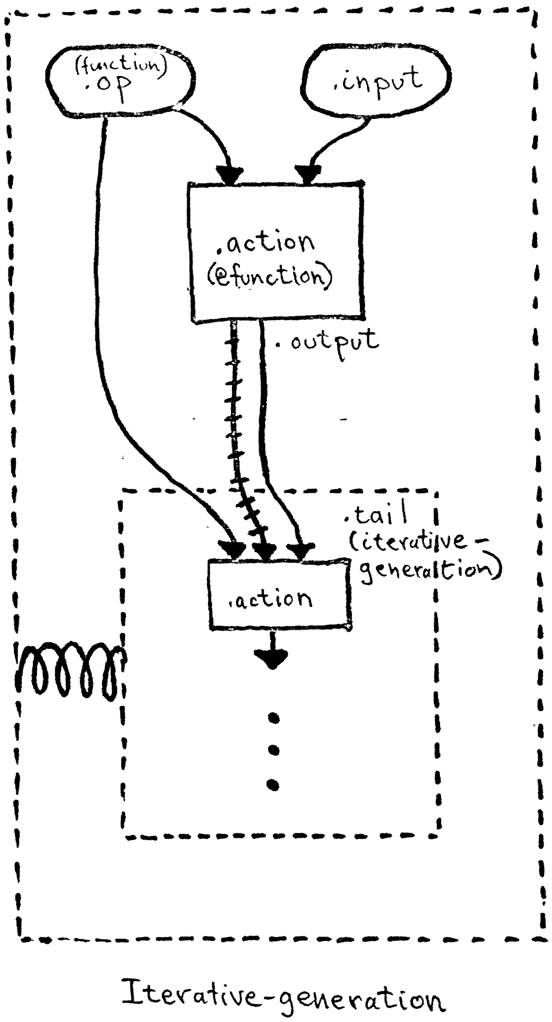


Figure 9-2. Iterative Generation Plan.

TemporalPlan **counting** *specialization* iterative-generation
roles .action(@function) .tail(counting)
constraints .action.op = oneplus ∧ .action.input = 1

The Iterative-application plan shown in Table 9-B and Fig. 9-3, captures the idea of repeatedly applying a given function to an input which is generated by some other part of the loop. The output of this application may then be the input to some other repeated computation. The application role in this plan is also called the Action.[1] For example in SEARCHLIST, the function CAR is applied to current value of L to get a new ENTRY each time around the loop.

```
(PROG (L ENTRY)
   LP (SETQ ENTRY (CAR L))
      ...
      (GO LP))
```

These two simple plans, Iterative-generation and Iterative-application, together with a number of common specializations, such as counting and CDRing, form the backbone of many common computations. For example, we have just analyzed the CAR-CDRing in SEARCHLIST as iterative generation and application. A similar programming cliché is looping through an array:

```
(PROG (I)
      ...
      (SETQ I 1)
   LP ...
      ...(ARRAYFETCH A I)...
      (SETQ I (1+ I))
      (GO LP)
      ...)
```

Here the iterative generation is counting and the application is fetching from the array.

The final iterative plan in Table 9-B is Iterative-filtering, also shown in Fig. 9-4. Typically it is used to select some subset of the values of a loop variable for special processing, as suggested by the following code.

```
(PROG (A)
      ...
   LP ...
      (COND ((P A) ...A...))
      ...
      (GO LP))
```

The non-recursive role in this plan, Filter, is a conditional structure (Cond). Each time around the loop, a given predicate (the same one each time) is used to test some object provided by the rest of the loop. Based on the result of this test, either the Then or the Else wings of the conditional will be executed.

1. Iterative-generation is in fact a specialization of Iterative-application, as can be seen in Table 9-B.

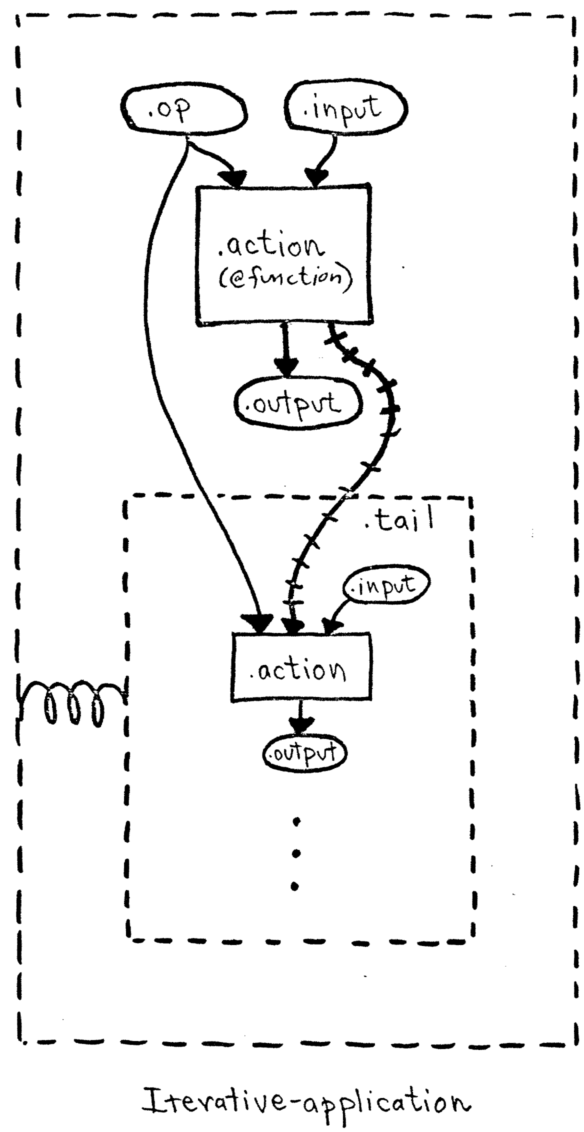


Figure 9-3. Iterative Application Plan.

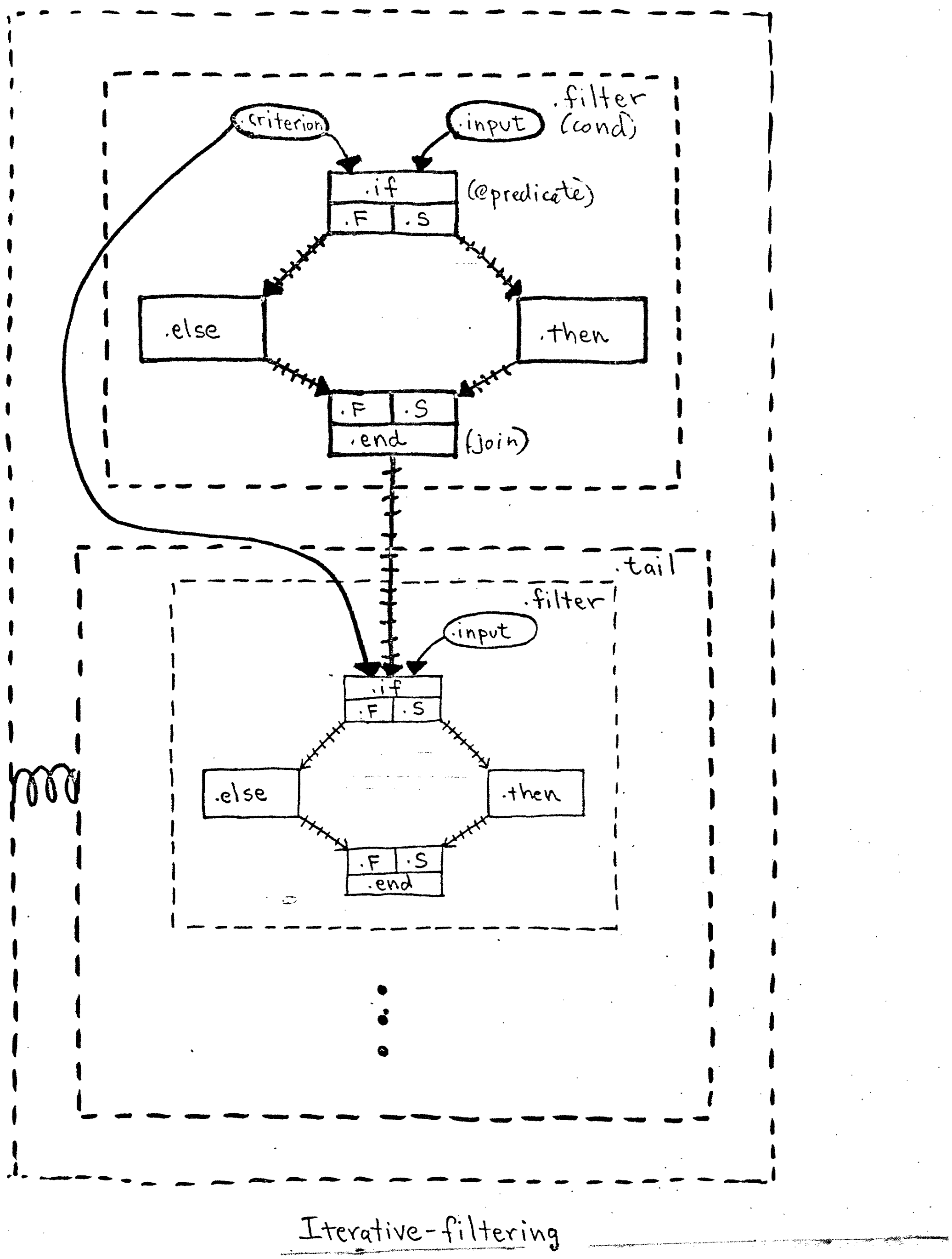


Figure 9-4. Iterative Filtering Plan.

Table 9-C. Iterative Termination.

TemporalPlan **iterative-termination** *extension* single-recursion
roles .exit(cond) .tail(iterative-termination+nil)
constraints [.tail = nil ↔ .exit.if.succeed≠⊥]
∧ cflow(.exit.if.fail,.tail.exit.if.in)
∧ cflow(.tail.exit.end.out,.exit.end.fail)

TemporalPlan **iterative-termination-predicate** *specialization* iterative-termination
roles .exit(cond) .tail(iterative-termination-predicate+nil)
constraints instance(@predicate,.exit.if)
∧ .exit.if.criterion = .tail.exit.if.criterion

TemporalPlan **iterative-termination-output** *specialization* iterative-termination
roles .exit(cond) .tail(iterative-termination-output+nil)
constraints instance(join-output,.exit.end)
∧ .tail.exit.end.output = .exit.end.fail-input

TemporalPlan **iterative-cotermination** *extension* iterative-termination-predicate
roles .exit(cond) .co-iterand(object) .tail(iterative-cotermination+nil)
constraints .co-iterand≠.exit.if.input

Type **iterative-termination+nil** *uniontype* iterative-termination nil

Type **iterative-termination-predicate+nil** *uniontype* iterative-termination-predicate nil

Type **iterative-termination-output+nil** *uniontype* iterative-termination-output nil

Type **iterative-cotermination+nil** *uniontype* iterative-cotermination nil

---

Let us now consider loops that have exits. The minimal plan for a loop with a single exit is Iterative-termination, shown in Table 9-C and Fig. 9-5. This plan describes loops with a single exit which are expected to terminate by that exit. It is similar to Iterative-filtering in that the non-recursive role, called Exit here, is an instance of Cond. In this plan, however, the recursive invocation (Tail) is constrained to occur *between* the test and the join. The succeed case of the test exits the loop by bypassing the recursive invocation; if the exit test fails, then the exit test of the tail must occur. Furthermore, if the tail of the loop exits through the end join, the whole loop ends. These control flow constraints, together with the constraints of Cond, prohibit other exits from the loop and require that the loop eventually terminates, i.e. it follows from the constraints on Iterative-termination that

cflow(.exit.if.in,.exit.end.out) .

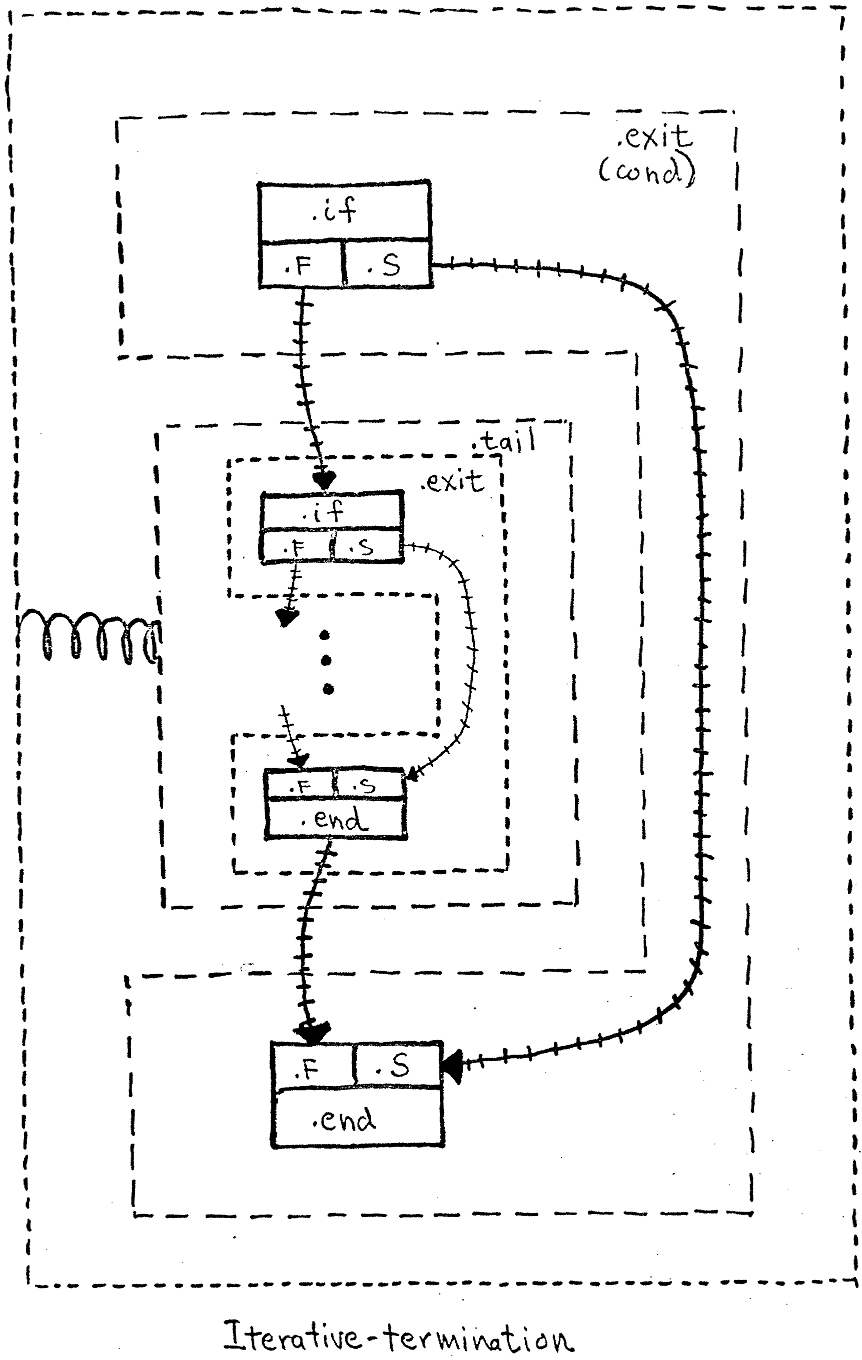


Figure 9-5. Iterative Termination Plan.

Table 9-D. Steady State Model.

TemporalPlan **iterative-steady-state** *extension* single-recursion
roles .step(situation) .tail(iterative-steady-state)
constraints cflow(.step,.tail.step)

TemporalOverlay **iterative-termination>steady-state:** iterative-termination → iterative-steady-state
correspondences
iterative-termination.exit.if.in = iterative-steady-state.step
∧ iterative-termination>steady-state(iterative-termination.tail) = iterative-steady-state.tail

---

Given an instance of Iterative-termination, the exit can be ignored for the purpose of steady state analysis by assuming that the exit test always fails. This modelling assumption is formalized by the overlay shown in Table 9-D. The Iterative-steady-state plan is introduced to represent a non-terminating loop, i.e. there is control flow from each Step of the iteration to the next. According to the overlay Iterative-termination>steady-state, an instance of Iterative-termination is viewed as an instance of Iterative-steady-state in which the current Step corresponds to the input situation of the exit test. The control flow constraint in the Iterative-steady-state plan thus amounts to assuming the exit test always fails.

The two specializations of Iterative-termination in Table 9-C concern what kind of test is performed and whether a final value is available as an output of the loop. Iterative-termination-predicate is the plan for the common case of loops where the exit tests a unary predicate that does not change as the computation proceeds. Iterative-termination-output is a fragment (used to build up other plans) which expresses the pattern of data flow needed in a loop to make the final value of some iterand available as an output of the entire loop when it is done. For such loops, the End join is an instance of Join-output, and the failure case of each join has data flow from the output of the End join of the tail. The final plan in this table, Iterative-cotermination, is also a fragment used to build up other plans. In this case, a Co-iterand role is added to Iterative-termination-predicate, which identifies an object of interest in the loop other than the input to the test.

Given these fragments, Iterative-search, shown in Table 9-E and Fig. 9-6, can be defined simply as a specialization of both Iterative-termination-predicate and Iterative-termination output. The only additional constraint added to express the idea of a search loop is that the output object is the final object that satisfied the predicate of the exit test. For example, in the SEARCHLIST program the value of ENTRY on the last repetition is returned.

```
(PROG (ENTRY)
   LP ...
      (COND ((FUNCALL P ENTRY)(RETURN ENTRY)))
      ...
      (GO LP))
```

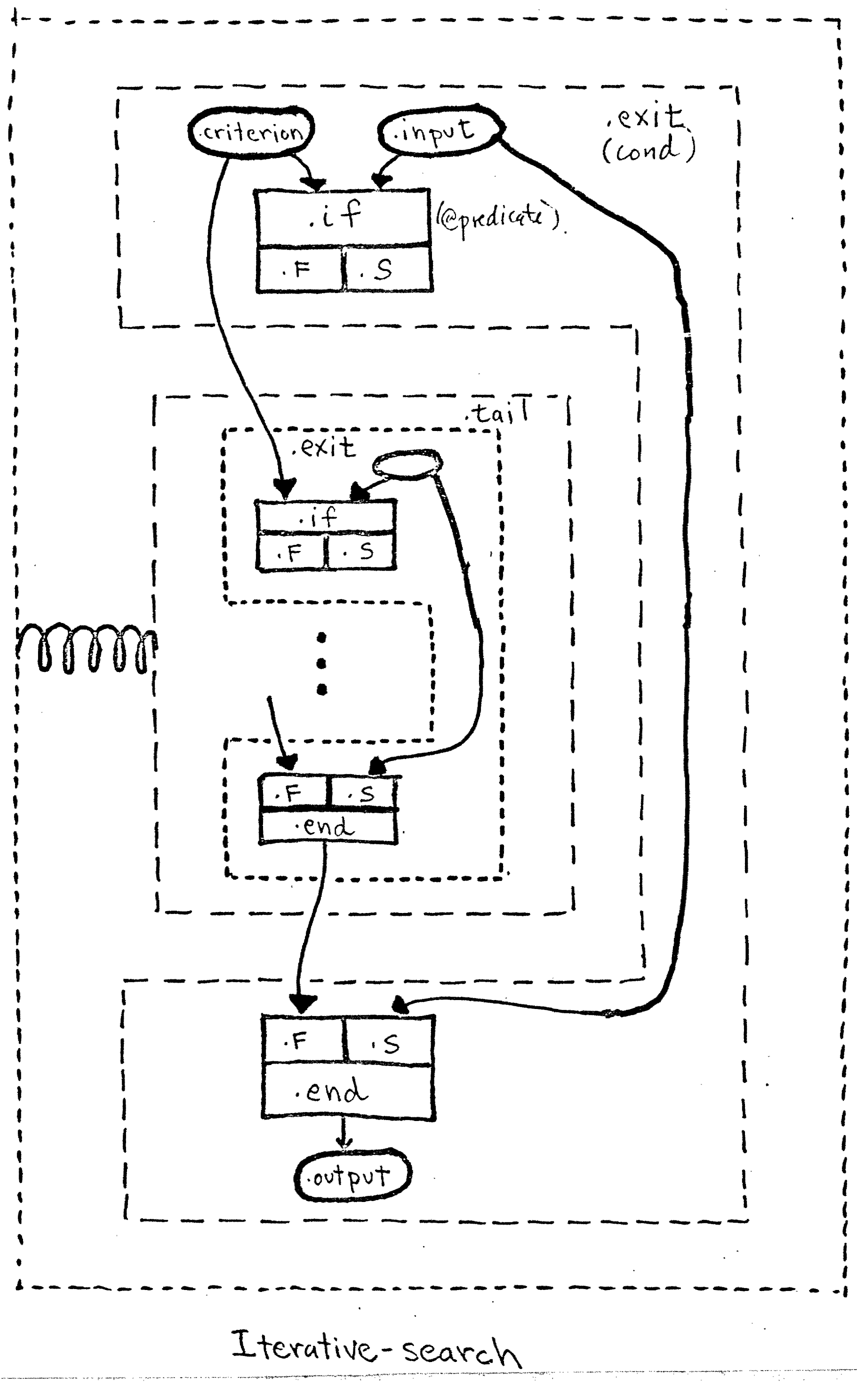


Figure 9-6. Iterative Search Plan.

Table 9-E. Iterative Search.

TemporalPlan **iterative-search**
specialization iterative-termination-predicate iterative-termination-output
roles .exit(cond) .tail(iterative-search+nil)
constraints .exit.if.input=.exit.end.succeed-input

TemporalPlan **iterative-cosearch** *specialization* iterative-cotermination
extension iterative-termination-output
roles .exit(cond) .co-iterand(object) .tail(iterative-cosearch+nil)
constraints .co-iterand$_{\text{.exit.if.in}}$=.exit.end.succeed-input

Type **iterative-search+nil** *uniontype* iterative-search nil

Type **iterative-cosearch+nil** *uniontype* iterative-cosearch nil

---

A closely related kind of search loop is one in which the object returned is not the same as the object tested in the exit test, as for example the program below, which calculates the length of a Lisp list.

```
(DEFINE LENGTH
   (LAMBDA (L)
      (PROG (N)
            (SETQ N 0)
         LP (COND ((NULL L)(RETURN N)))
            (SETQ L (CDR L))
            (SETQ N (1+ N))
            (GO LP))))
```

Here consecutive values of L (generated by CDR) are tested by NULL, while at the same time N is counting. When the NULL test finally succeeds, the current value of N is returned as the output of the loop. This pattern of loop termination is formalized as the Iterative-cosearch plan shown in Table 9-E. This plan is a specialization of Iterative-cotermination and an extension of Iterative-termination-output (adding the Co-iterand role), in which the output object is constrained to be the final co-iterand when the loop exits.

A third basic loop plan which returns an output is Iterative-accumulation, shown in Table 9-F and Fig. 9-7. On each repetition of an accumulation loop, an operation (Add) is performed which takes an Old object and another input, and returns a New object of the same type. Set-add and Push are examples

---

Table 9-F. Iterative Accumulation.

TemporalPlan **iterative-accumulation** *extension* iterative-termination-output
roles .exit(cond) .init(object) .add(old+input+new) .tail(iterative-accumulation)
constraints .init=.add.old ∧ .init=.exit.end.succeed-input
∧ .add.new=.tail.init ∧ cflow(.exit.if.fail,.add.in)
∧ cflow(.add.out,.tail.exit.if.in) ∧ ∧ cflow(.tail.end.out,.end.fail)

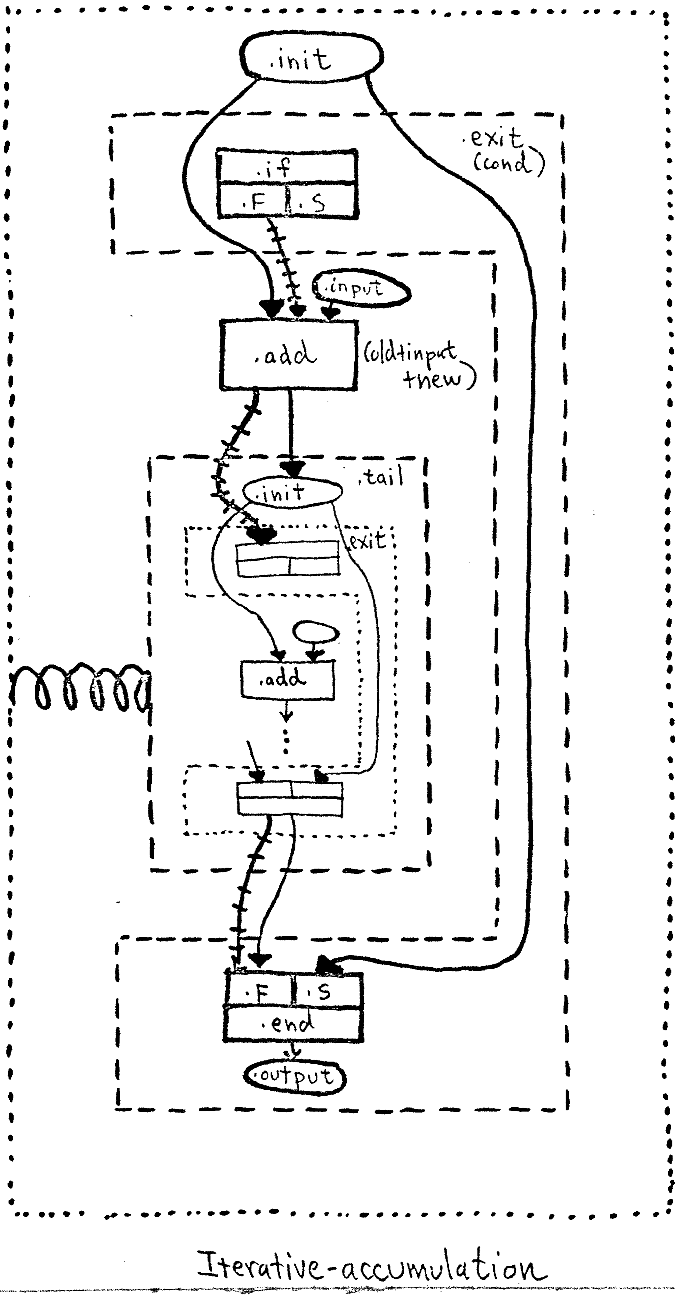


Figure 9-7. Iterative Accumulation Plan.

of such operations for sets and lists. The Old input on the first repetition is called the Init; on successive repetitions of the loop, the Old input of each Add is the same as the New output of the previous Add. The Input of Add on each repetition is provided from the rest of the loop. For example, the following is code for a typical accumulation loop.

```
(PROG (ACCUM ...)
      (SETQ ACCUM ...)
      ...
   LP (COND ((...)(RETURN ACCUM)))
      ...
      (SETQ ACCUM (CONS ... ACCUM))
      ...
      (GO LP))
```

Here the Add operations are instances of Push (implemented as CONS for Lisp lists). The value of ACCUM returned from the loop (Exit.End.Output) is the same as the New output of the last Push, except when the loop exits the first time, when the value of ACCUM is determined by the initial SETQ (Init).

Specializations of Iterative-accumulation for different types of Add and Init correspond to common programming cliché. If the Add roles are filled by instances of Push and the Init is Nil, then we are accumulating the Input's as a list. If the Add roles are filled by instances of Set-add and the Init is an empty set, then we are accumulating the Input's as a set. Applications of Plus and Times can also be viewed as instances of Old+input+new.[1] With appropriate Init's, 0 and 1 respectively, these accumulation loops compute the sum and product of the Input's. These ideas about accumulation will be formalized further later in this chapter.

Table 9-G. Two Exit Loops.

TemporalPlan **cascade-iterative-termination** *extension* single-recursion
roles .if-one(test) .if-two(test) .end-one(join) .end-two(join)
.tail(cascade-iterative-termination)
constraints cflow(.if-one.fail,.if-two.in) $\wedge$ cflow(.if-one.succeed,.end-one.succeed)
$\wedge$ cflow(.if-two.fail,.tail.if-one.in) $\wedge$ cflow(.if-two.succeed,.end-two.succeed)
$\wedge$ cflow(.tail.end-one.out,.end-one.fail) $\wedge$ cflow(.tail.end-two.out,.end-two.fail)
$\wedge$ (.tail = nil $\leftrightarrow$ (.if-one.succeed $\neq \perp$ $\vee$ .if-two.succeed $\neq \perp$))
$\wedge$ (.if-one.in $\neq \perp$ $\leftrightarrow$ (.end-one.out $\neq \perp$ $\vee$ .end-two.out $\neq \perp$))

TemporalOverlay **cascade>iterative-termination:**
cascade-iterative-termination $\rightarrow$ iterative-termination
correspondences
cascade-iterative-termination.if-one = iterative-termination.exit.if
$\wedge$ cascade-iterative-termination.end-one = iterative-termination.exit.end
$\wedge$ cascade>iterative-termination(cascade-iterative-termination.tail) = iterative-termination.tail

---

1. See overlay @Binfunction>old + input + new in the appendix.

The minimal plan for two exits from a loop is Cascade-iterative-termination shown in Table 9-G and Fig. 9-8. For example, the loop in the LOOKUP procedure of Chapter Five had two exits as shown below.

```
(PROG (...)
       ...
    LP (COND ( ... (RETURN ...)))
       ...
       (COND (... (RETURN ...)))
       ...
       (GO LP))
```

The plan for this code has two tests, If-one and If-two, and two corresponding joins, End-one and End-two. Each time around the loop, If-one is performed first; if it fails, then If-two is performed. If the second test fails, the loop is repeated. If either test succeeds, the loop is terminated by control flow to the corresponding join which bypasses the recursive invocation (Tail). The final constraint on Cascade-iterative-termination says that if the input situation of the first test occurs, then one of the output situations of the joins will occur. This means that the loop is expected to eventually terminate on one of its exits.

Steady state analysis of two exit loops is achieved in two steps. First a two exit loop is viewed as a single exit loop by assuming the second exit is never taken (using the overlay Cascade>iterative-termination in Table 9-G). The first exit can then be ignored using the Iterative-termination>steady-state overlay defined earlier, as for any other single exit loop.

These few basic patterns of iterative computation, termination, application, generation, accumulation and filtering appear over and over again in routine programming. In fact, many recursive programs are built out of nothing but these plans. Waters [73] did an analysis of 44 programs chosen at random from the 220 programs which comprise the IBM Fortran Scientific Subroutine Package. All of the 164 loops contained in these programs could be analyzed solely in terms of these basic patterns.[1] Furthermore, most of these were instances of a small number of common specializations of the basic plans. Out of a total of 370 instances of generation and accumulation, 82 percent were either summation, product aggregation, maximum, minimum, or counting.[2] Out of a total of 186 loop exit tests, 89 percent were simple comparisons with a fixed number.[3]

Given that we have identified instances of these standard recursive plans in a program, the question remains of how to represent the connection between, say, an generation and an application. Temporally, the components of each are interleaved, but it seems more logical to view the generation and application as being composed in some way. The next section shows how to formalize this viewpoint.

---

1. Several loops had more than two exits, but this is a straightforward generalization of the one and two exit plans presented here.
2. Waters' analysis does not distinguish between generation and accumulation. They are both categorized as augmentations (application of a function or binary function) with feedback.
3. The input being tested in most of these exit tests was a simple sequence of numbers, most often just counting up from one. Thus for most of these loops termination is obvious. This is typical of routine programming — in most cases the question of termination is settled by recognition of well-known patterns.

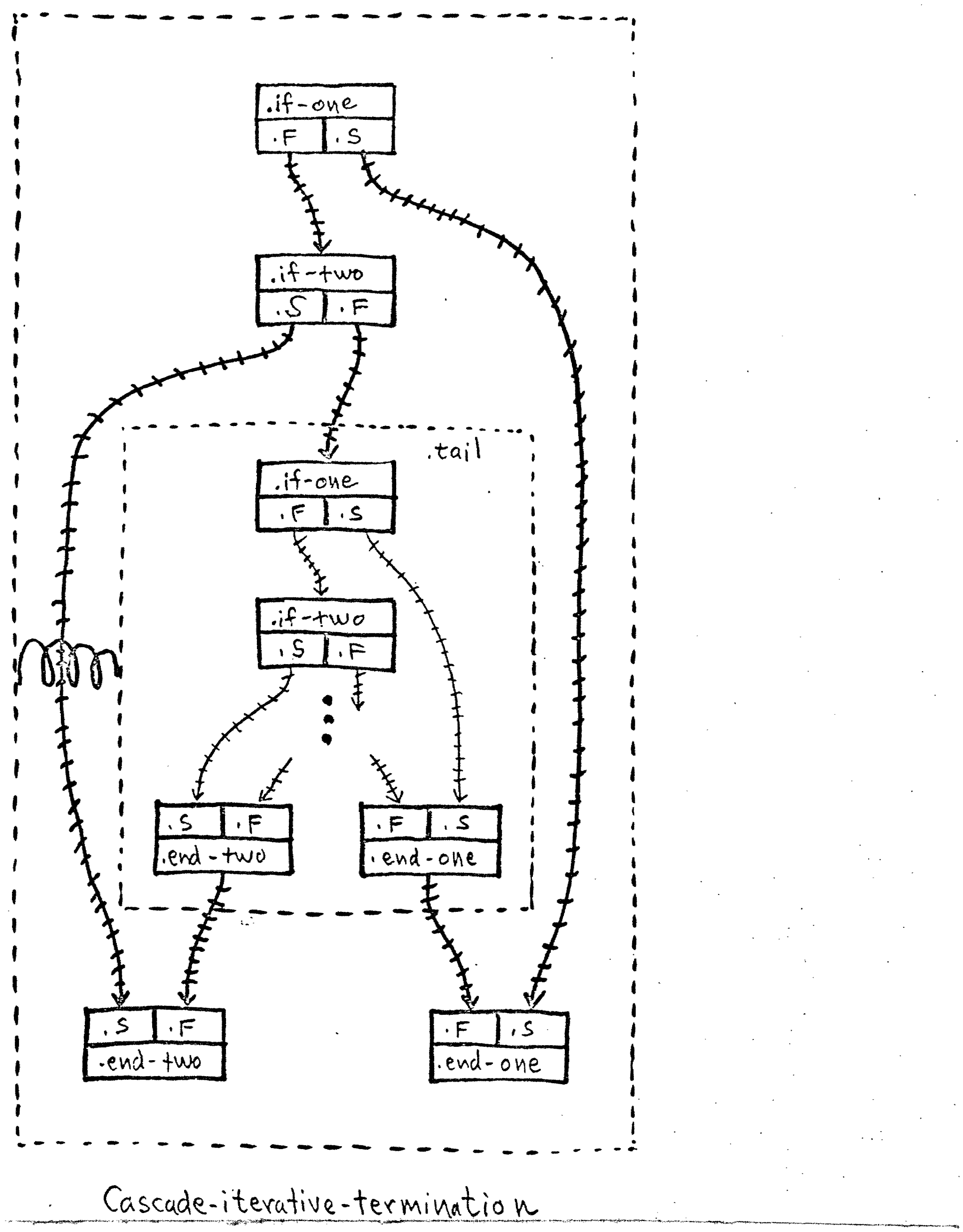


Figure 9-8. Two Exit Loop Plan.

9.3 Temporal Abstraction

The basic idea of temporal abstraction is to view all the objects which fill a given role in a recursive temporal plan as a single data structure.[1] In terms of Lisp code, this often corresponds to having an explicit representation for the history of values taken on by a particular variable at a particular point in a loop or other recursive program. For example, in the example program for searching a list introduced earlier, we would like to talk about the values of L at the underlined point each time around the loop.

```
(DEFINE SEARCHLIST
   (LAMBDA (L P)
      (PROG (ENTRY)
         LP (SETQ ENTRY (CAR L))
            (COND ((FUNCALL P ENTRY)(RETURN ENTRY)))
            (SETQ L (CDR L))
            (GO LP))))
```

In general, temporal abstraction gives rise to tree structures. In this section, however, we will discuss only loops, which give rise to linear structures. Temporal abstraction is formalized using overlays. The left side of such an overlay is a recursive temporal plan; the right side is a recursive data plan of the same order (i.e. they are both singly recursive, or doubly, etc.). The definition of the overlay establishes a correspondence between roles in the recursive temporal plan and roles in the data plan, such that the time behavior of a computation which is an instance of the plan on the left is abstracted as a data structure which is an instance of the plan on the right.

Stream Overlays

In the case of loops, temporal abstraction amounts to thinking of a program in terms of *streams*. Streams at particular points in a loop are chosen for temporal abstraction based the analysis of the loop according the taxonomy of iterative temporal plans (generation, application, filtering, etc.) introduced earlier. For example, the temporal abstraction of the underlined values of L in the SEARCHLIST program above is the stream of objects generated by iterative CDR generation. The overlay below and in Fig. 9-9 shows how to express this abstraction formally.

TemporalOverlay **generation-stream:** iterative-generation → list
correspondences iterative-generation.action.input = list.head
∧ generation-stream(iterative-generation.tail) = list.tail

The head of the list corresponds to the input of the action of the iterative generation; the tail of the list is recursively defined as the temporal abstraction of the tail of the generation.

We next discuss how to abstract the temporal behavior of Iterative-application. For example, we would like to express the relationship in the code below between the values of L and the values of ENTRY at the underlined points each time around the loop.

---

1. Both Shrobe [64] and Waters [73] use the idea of temporal abstraction, but with slightly different formalizations than presented here.

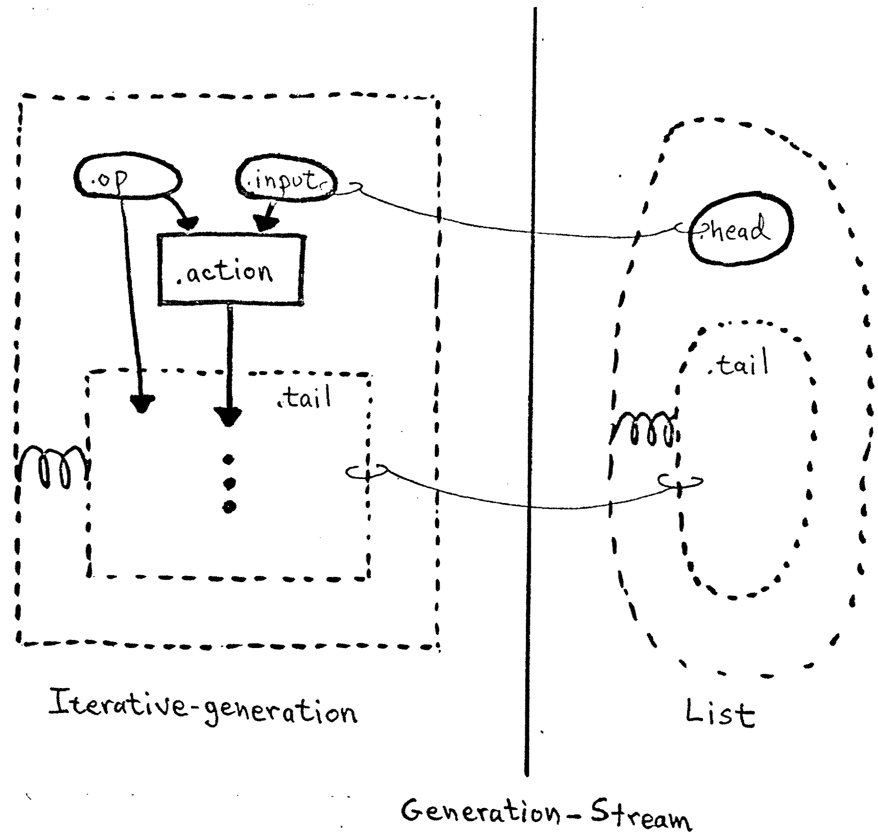


Figure 9-9. Stream Abstraction of Iterative Generation.

```
(DEFINE SEARCHLIST
   (LAMBDA (L P)
      (PROG (ENTRY)
         LP (SETQ ENTRY (CAR L))
            (COND ((FUNCALL P ENTRY)(RETURN ENTRY)))
            (SETQ L (CDR L))
            (GO LP))))
```

This is achieved by defining two overlays, shown in Table 9-H and Fig. 9-10, which temporally abstract the input and output roles of the Action of the iterative application. The relationship between these two streams is then most conveniently expressed by viewing them as sequences, as explained in the next section.

Temporal Sequences

Streams viewed as sequences are called temporal sequences. Making this abstraction step allows us to use the input-output specifications on sequences to describe the relationship between changing values in loops. In particular, iterative application can be thought of as implementing a Map operation from the stream of inputs of the Action role (viewed as a sequence) to the stream of outputs. This is expressed as the Temporal-map overlay shown in Table 9-I and in Fig. 9-11.

On the right side of this overlay is the Map plan. The inputs to the Action of the iterative application (e.g. the values of L above) are abstracted as the input to Map. The outputs of the Action (e.g. the values of ENTRY above) are modelled as the output of Map. The Op of Map corresponds to the Op of the Action. What we are doing in this overlay is modelling the time behavior of the recursive temporal plan on the left as a single step in some other time domain represented on the right.

Iterative generation can be similarly abstracted as an Iterate operation, as shown in Table 9-I. The input to the operation is an iterator whose Op is the Op of the Action of the generation (viewed as a relation) and whose Seed is the initial Input (according to the Temporal-iterator overlay). The output sequence is the generated stream, as defined in the preceding section, viewed as a sequence.

Finally, note the following property of Generation-stream which ties together two different viewpoints on generation loops that have been introduced in this chapter: if a generation loop, viewed as an iterator, generates a thread, then the temporally abstracted (irredundant) list of inputs to the action of that loop, viewed as a thread, is the same thread as generated by the iterator.

Table 9-H. Application Stream Overlays.

TemporalOverlay **application-in-stream:** iterative-application → list
correspondences iterative-application.action.input = list.head
∧ application-in-stream(iterative-application.tail) = list.tail

TemporalOverlay **application-out-stream:** iterative-application → list
correspondences iterative-application.action.output = list.head
∧ application-out-stream(iterative-application.tail) = list.tail

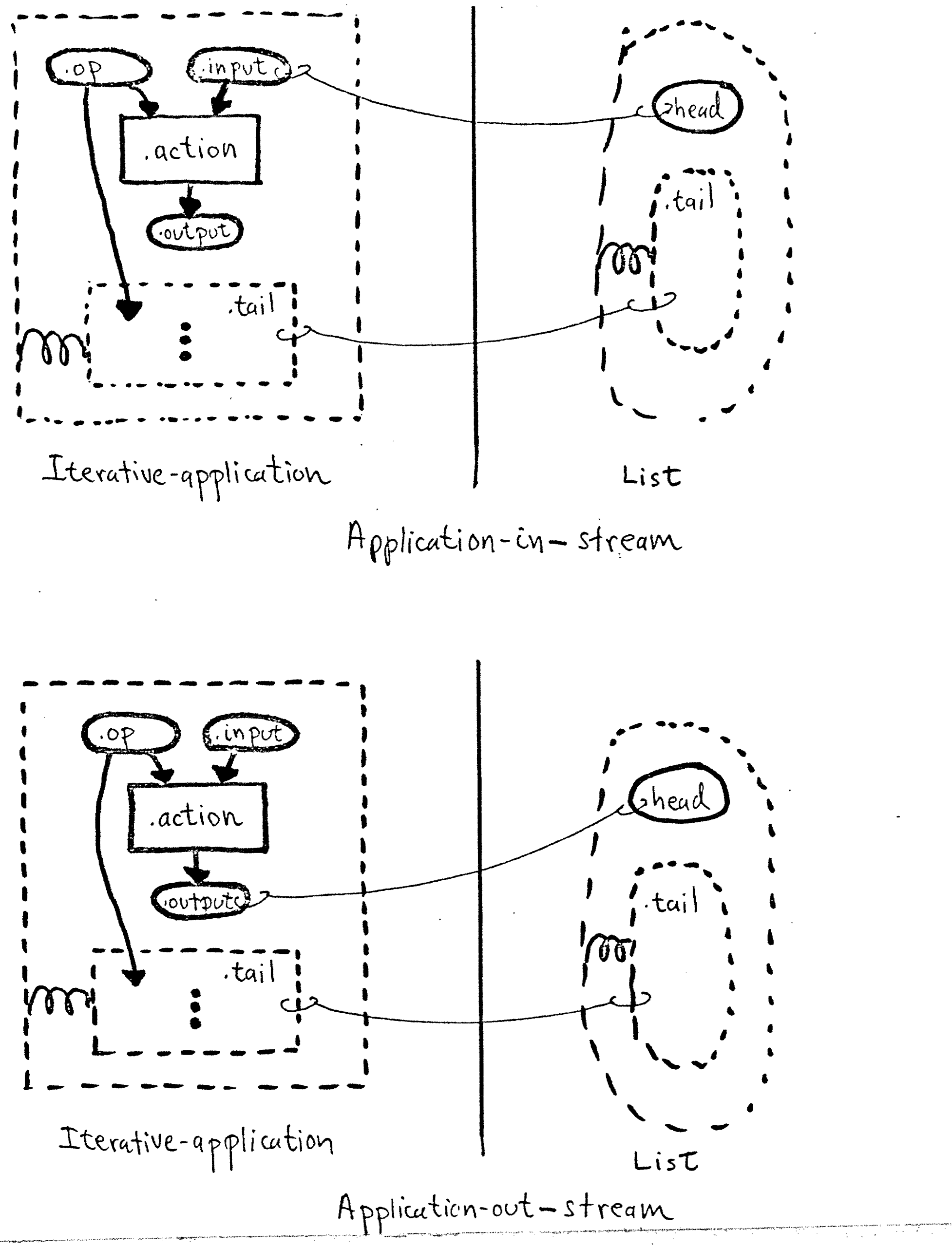


Figure 9-10. Stream Abstractions of Iterative Application.

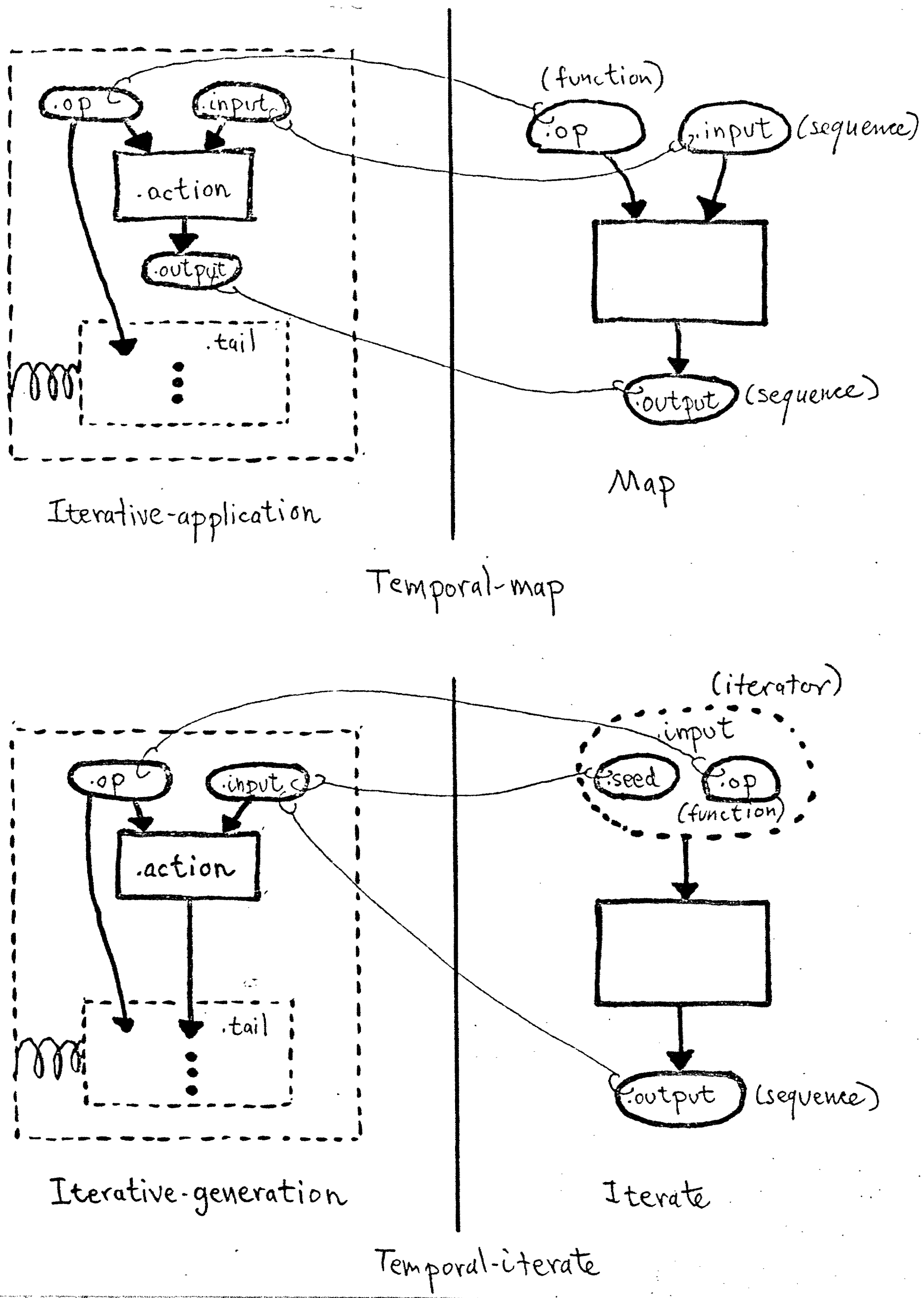


Figure 9-11. Temporal Overlays for Application and Generation.

Table 9-I. Temporal Sequence Overlays.

TemporalOverlay **temporal-map**: iterative-application → map
correspondences
iterative-application.action.op = map.op
∧ list>sequence(application-in-stream(iterative-application)) = map.input
∧ list>sequence(application-out-stream(iterative-application)) = map.output

TemporalOverlay **temporal-iterate**: iterative-generation → iterate
correspondences
temporal-iterator(iterative-generation) = iterate.input
∧ list>sequence(generation-stream(iterative-generation)) = iterate.output
∧ iterative-generation.action.in = iterate.in

TemporalOverlay **temporal-iterator**: iterative-generation → iterator
correspondences iterative-generation.action.input = iterator.seed
∧ function>binrel(iterative-generation.action.op) = iterator.op

---

TemporalOverlay **generation-stream**: iterative-generation → list
properties ∀ *I* [instance(thread,generator>digraph(temporal-iterator(*I*)))
⊃ ∀ *s* list>thread(generation-stream(*I*),*s*) = generator>digraph(temporal-iterator(*I*),*I*.action.in)]

Temporal Data Flow

In this section we further develop the notion of stream overlays in order to specify the connection between the temporal abstractions of different parts of a loop. For example, in the SEARCHLIST example, shown again below, the stream generated by the iterative CDR generation is the same as the input stream to the iterative CAR application.

```
(DEFINE SEARCHLIST
   (LAMBDA (L P)
      (PROG (ENTRY)
         LP (SETQ ENTRY (CAR L))
            (COND ((FUNCALL P ENTRY)(RETURN ENTRY)))
            (SETQ L (CDR L))
            (GO LP))))
```

This means that data flow between operations in the recursive view implies data flow between operations in the temporally abstracted view.

For example, Fig. 9-12 shows the temporal sequence analysis of SEARCHLIST. On the left is the unanalyzed recursive computation. As has been pointed out before, this diagram contains an instance of Iterative-generation (the Action role corresponds to role Two of the surface plan), of Iterative-application (the Action role corresponds to role One of the surface plan), and of Iterative-search (the Exit.If role corresponds to the If role of the surface plan). The right side of the figure shows the plan after recognizing these iterative patterns and applying the Temporal-iterate, Temporal-map, and Temporal-

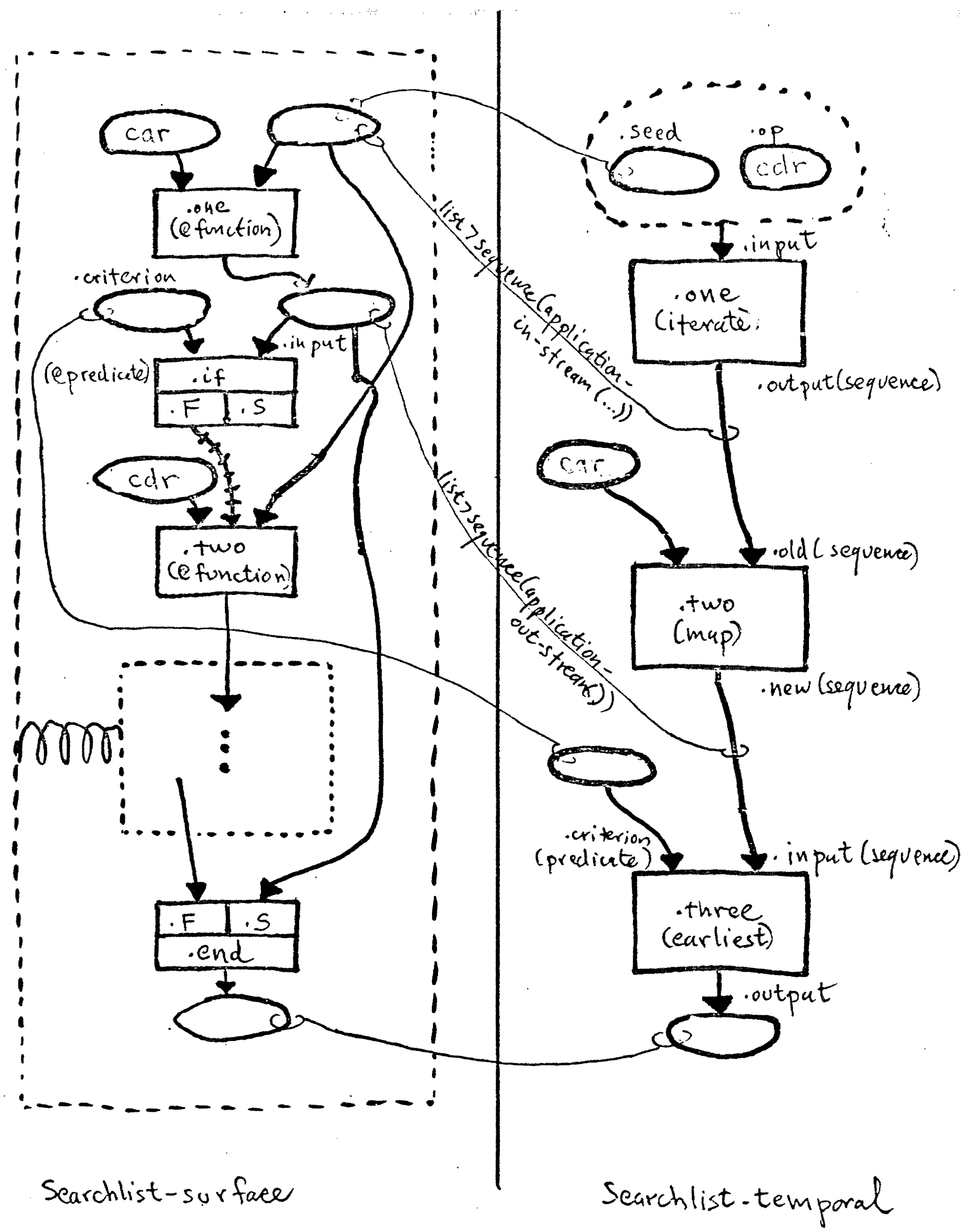


Figure 9-12. Temporal Analysis of SEARCHLIST.

earliest overlays.[1] Temporal sequence abstractions are labelled. The objects generated by CDR are temporally abstracted as the output sequence of One (Iterate) on the right. Furthermore, since the input to the Action of the generation on each repetition is the same as the input to the Car application, this sequence is therefore also the temporal abstraction of the inputs to the iterative application. Similarly, data flow between the output of the Action of iterative application and the input to the exit test of iterative search on each repetition in the surface plan implies data flow from the output of the Two (Map) to the input of Three (Earliest) in the temporal view.

Thus we see that temporal analysis leads to a viewpoint on loops in which there are producers, transducers and consumers of streams. An overlay like Temporal-iterate models a part of a loop which produces a stream; Temporal-map models a part of a loop which consumes one stream and produces another (i.e. a transducer); and Temporal-earliest models a stream consumer.

The pattern of Iterate and Map in SEARCHLIST (particularly implemented temporally as iterative generation composed with application) is a common one. This plan is called List-generation, as shown in Table 9-J, because the output sequence of the Map operation (role Two), viewed as a list, is the same as the labelled thread whose spine is generated by the input to the Iterate operation, and whose label is the

---

Table 9-J. List Generation.

TemporalPlan **list-generation**
  properties $\forall P$ [instance(list-generation,P) $\supset$
    $\forall s$ list>sequence(generation>list(P),s) = sequence(P.two.output,P.two.out)]
  roles .one(iterate) .two(map)
  constraints .one.output = .two.input $\wedge$ cflow(.one.out,.two.in)

TemporalOverlay **generation>list**: list-generation $\rightarrow$ list
  definition L = generation>list(P) $\equiv$
    $\exists T$ [instance(labelled-thread,T)
    $\wedge$ digraph(T.spine,P.one.in) = generator>digraph(P.one.input,P.one.in)
    $\wedge$ function(T.label,T.two.out) = function(P.two.op,P.two.out)]

TemporalPlan **car+cdr** *specialization* list-generation
  properties $\forall P$ [instance(car+cdr,P) $\supset$ dotted-pair>list(P.one.input.seed,P.one.in) = generation>list(P)]
  roles .one(iterate) .two(map)
  constraints instance(cdr-iterator,.one.input) $\wedge$ .two.op = car

DataPlan **cdr-iterator** *specialization* iterator
  roles .seed(dotted-pair) .op(many-to-one)
  constraints .op = function>binrel(cdr)

---

1. The overlay between Iterative-search and Earliest will be defined later in this section. To simplify the presentation, the figure omits several intermediate analysis steps.

function applied in the Map operation. This relationship is expressed by the overlay Generation>list, also shown in Table 9-J.

Car+cdr is the common special case of Lisp list generation, wherein the input to Iterate is an instance of Cdr-iterator (an iterator in which the Op is Cdr) and the function applied by Map is Car. It thus follows that the list generated by Car+cdr, according to the overlay Generation>list, is the same as the list implemented by the dotted pair which is the seed of the iterator input to Iterate, according to the overlay Dotted-pair>list.

Termination

We move on now to the temporal abstraction of termination plans, in particular the overlays in Table 9-K and Fig. 9-13, which express how to view the inputs to the exit test of Iterative-termination-predicate as a finite list.

The basic form of the Termination-in-stream overlay is the same as Application-in-stream, i.e. the head of the list is the input on the first repetition and the tail of the list is defined recursively. In this overlay however, the recursive definition is split into two cases:[1] if the exit test succeeds, then the tail of the list is Nil; if it fails, then the tail of the list is the temporal abstraction of the tail of the termination plan. This means that the temporal abstraction of the inputs to a termination which exits on the first step

---

Table 9-K. Termination Stream Overlays.

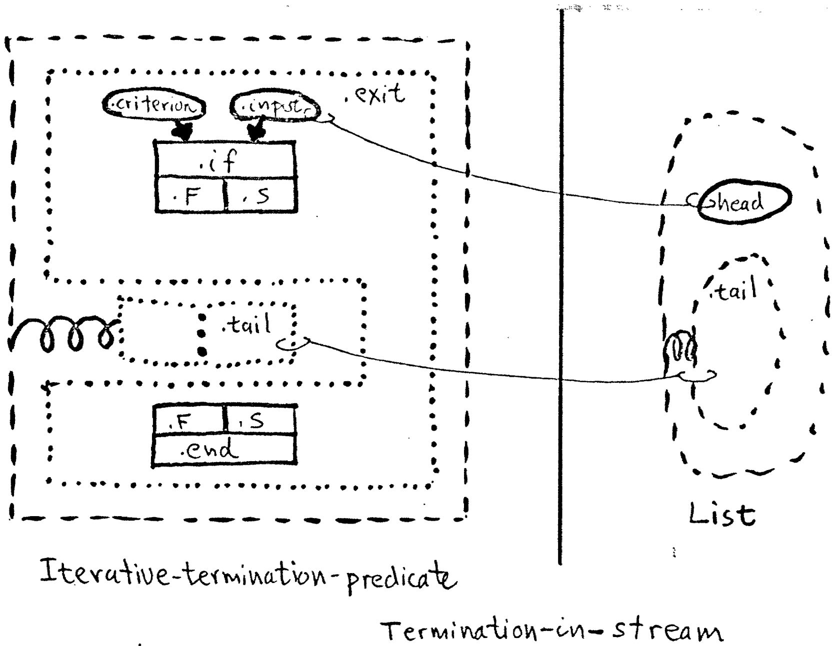
TemporalOverlay **termination-in-stream:** iterative-termination-predicate → finite-list
correspondences
iterative-termination-predicate.exit.if.input = finite-list.head
∧ (*if* iterative-termination-predicate.tail = nil
then nil
else termination-in-stream(iterative-termination-predicate.tail))
= finite-list.tail

TemporalOverlay **termination-fail-stream:** iterative-termination-predicate → finite-list+nil
definition L = termination-fail-stream(T) ≡
∀s [list>sequence(L,s) = butlast(termination-in-stream(T))]

TemporalOverlay **steady-state-stream:** iterative-termination-predicate → list
correspondences
iterative-termination-predicate.exit.if.input = list.head
∧ steady-state-stream(iterative-termination>steady-state(iterative-termination-predicate.tail))
= list.tail

---

1. The standard abbreviated form A = (*if* X *then* Y *else* Z) expands into [X ⊃ A = Y] ∧ [¬X ⊃ A = Z].



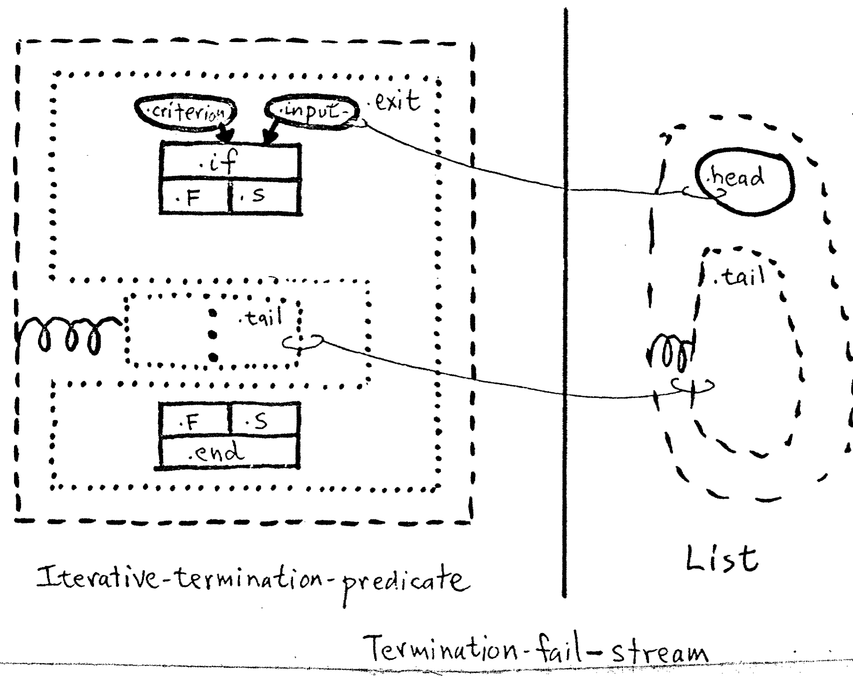


Figure 9-13. Stream Abstractions of Iterative Termination.

is a singleton list (one with a Nil tail), and that for all uses of Termination-in-stream the last object in the list (i.e. the head of the sublist with the Nil tail) satisfies the exit predicate.

Termination-fail-stream is the overlay which abstracts the inputs to the exit test of a loop as seen in an environment where the test is known to have failed. For example, the difference between Termination-in-stream and Termination-fail-stream is the difference between the stream of values of L seen at the first underlined point below versus the second.

```
(PROG (L)
      ...
   LP ...L...
      (COND ((P L)(RETURN ...)))
      ...L...
      (GO LP))
```

The Termination-fail stream is defined in terms of the Termination-in stream. It is the same as the Termination-in stream, except one term shorter (this is expressed formally in terms of sequences and the Butlast function).

Like Iterative-application, Iterative-termination-predicate describes a fragment of a loop which has some of its inputs provided by other parts of the loop. The relationship between a termination test and the other parts of the loop is most conveniently expressed in terms of the relationship between the Termination-in or Termination-fail streams and the stream of inputs to the test in the steady state analysis (see Table 9-D) The stream of inputs to the test in the steady state analysis is specified by the overlay Steady-state-stream in Table 9-K. It has a recursive definition similar to the other stream overlays for iterative termination. In this case, however, we are talking about the stream of inputs which would be seen in the input situation of the exit test in a non-terminating loop.

The relationship between the steady state stream and the Termination-in and Termination-fail streams is most conveniently expressed in terms of Truncate and Truncate-inclusive operations on temporal sequences. For example, in the following loop the temporal sequence of values of N at the underlined point under the steady state assumption is 1,2,3,..., as generated by Natural-iterator (an iterator in which the Op is Oneplus and the Seed is 1). The effect of adding the termination test is to truncate this sequence at 10 (inclusively at the point indicated).[1]

```
(PROG (N)
      (SETQ N 1)
   LP ...N...
      (COND ((= N 10)(RETURN ...)))
      (SETQ N (1+ N))
      (GO LP))
```

The overlays in Table 9-L formalize this analysis. Let us first consider Temporal-truncate-inclusive with reference to Fig. 9-14, which shows its application to the example program above. On the left is the unanalyzed surface plan. In the center is the steady state analysis in which an instance of Iterative-

1. In a later section, an overlay will be introduced which captures the relationship between using N at the point indicated and a loop that checks for (= N 11) and uses N after the test.

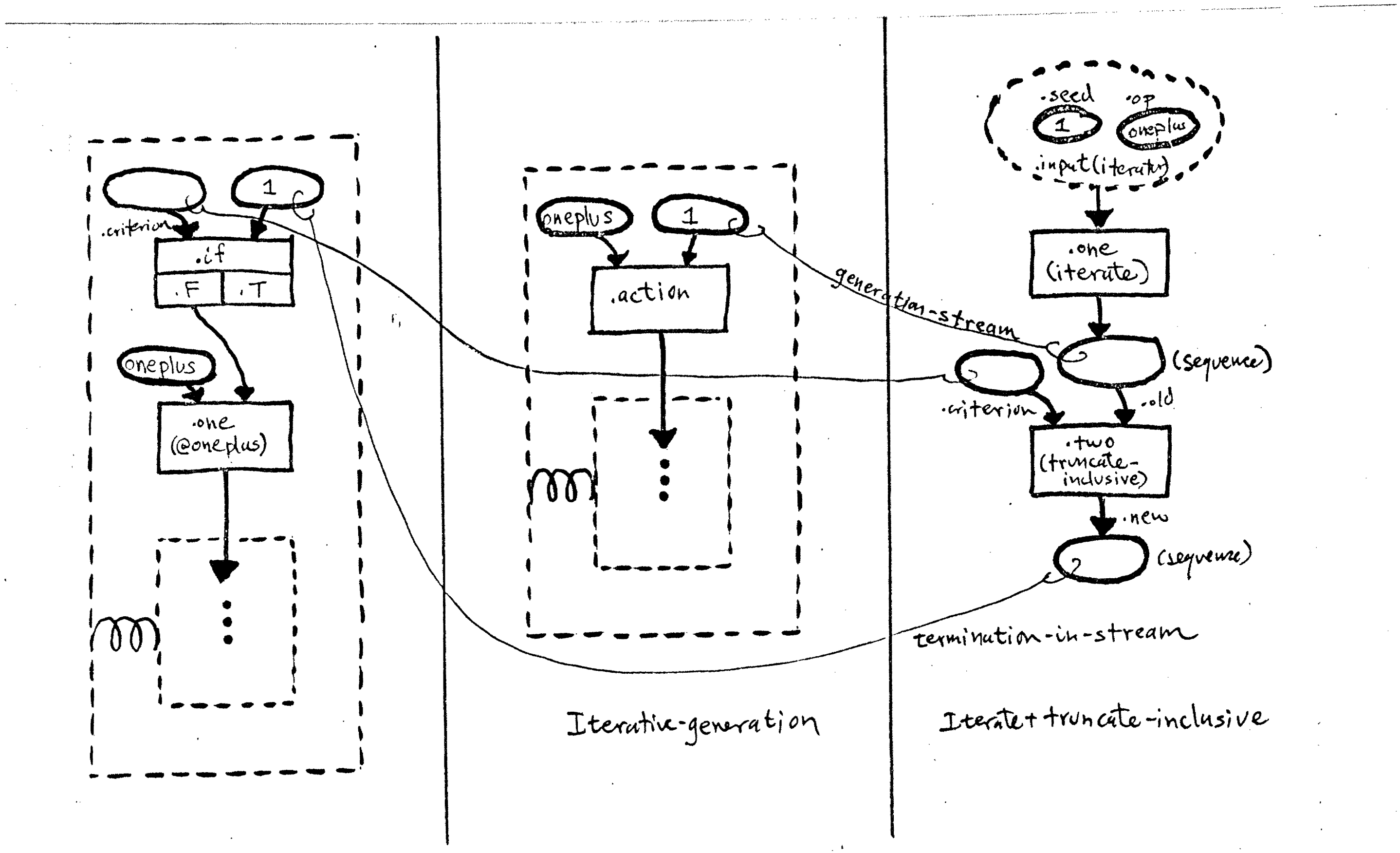


Figure 9-14. Counting Program at Various Levels of Abstraction.

Table 9-L. Temporal Truncation.

TemporalOverlay **temporal-truncate-inclusive:** iterative-termination-predicate → truncate-inclusive
correspondences
iterative-termination-predicate.exit.if.criterion = truncate-inclusive.criterion
∧ list>sequence(steady-state-stream(iterative-termination-predicate)) = truncate.input
∧ list>sequence(termination-in-stream(iterative-termination-predicate))
= truncate-inclusive.output
∧ iterative-termination-predicate.exit.end.out = truncate-inclusive.out

TemporalOverlay **temporal-truncate:** iterative-termination-predicate → truncate
correspondences
iterative-termination-predicate.exit.if.criterion = truncate.criterion
∧ list>sequence(steady-state-stream(iterative-termination-predicate)) = truncate.input
∧ list>sequence(termination-fail-stream(iterative-termination-predicate)) = truncate.output
∧ iterative-termination-predicate.exit.end.out = truncate.out

---

generation has been recognized. The temporal sequence generated by this generation is abstracted on the right as the output of an Iterate operation. This sequence is then the input to a Truncate-inclusive operation whose output is the sequence of values of N actually seen temporally at the underlined point. Note that the termination constraint on the Iterative-termination plan and its specializations is consistent with the precondition on Truncate that there exists a term of the sequence which satisfies the given criterion. The Temporal-truncate overlay is similar to Temporal-truncate-inclusive, except that the input sequence is the Termination-fail sequence, rather than the Termination-in sequence.

Table 9-M. Iterate and Truncate.

TemporalPlan **iterate+truncate**
roles .one(iterate) .two(in+out)
constraints instance(irredundant-sequence,.one.output)
∧ [instance(truncate,.two) ∨ instance(truncate-inclusive,.two)]
∧ .one.output = .two.input ∧ cflow(.one.out,.two.in)

TemporalOverlay **iterate+truncate>truncated-thread:** iterate+truncate → truncated-thread
properties ∀*IT* [*T* = iterate+truncate>truncated-thread(*I*) ⊃
[[instance(truncate,*I*.two)
⊃ ∀*s* truncated>digraph(*T*,*s*) = sequence>thread(*I*.two.output,*I*.two.out)]
∧ [instance(truncate-inclusive,*I*.two)
⊃ ∀*s* truncated>digraph-inclusive(*T*,*s*) = sequence>thread(*I*.two.output,*I*.two.out)]]
correspondences
sequence>thread(iterate+truncate.one.output) = truncated-thread.base
iterate+truncate.two.criterion = truncated-thread.criterion

The pattern of Iterate and Truncate shown in Fig. 9-14, particularly implemented temporally as a loop in which the termination directly tests the current value of an iterative generation, is a common one. The Iterate+truncate plan shown in Table 9-M expresses this in a data flow constraint between the output sequence of the Iterate operation (One) and the input sequence of the Truncate(-inclusive) operation (Two).

Iterate+truncate can be further abstracted as a truncated thread, as described by the overlay in Table 9-M. The base of the truncated thread is the irredundant sequence output of the Iterate operation, viewed as a thread; the criterion is the criterion of the Truncate(-inclusive) operation. Note the property of this overlay, which expresses the relation between the sequence and directed graph views of these operations. In particular, the finite graph implemented (inclusively) by the truncated thread is the same as the output of the Truncate(-inclusive) operation, viewed as a thread.

Another temporal abstraction involving termination is to view an iterative search loop as the implementation of an Earliest operation, as shown in Table 9-N and Fig. 9-15. In this overlay, the input sequence to the Earliest operation is the steady state stream of inputs to the test of the iterative search, viewed as a sequence. The output of the ending join of the search plan is the output of the Earliest operation. Note that the termination constraint of Iterative-search is consistent with the precondition on Earliest which states that there exists a term of the input sequence which satisfies the given criterion.

Cotermination loops are temporally abstracted using overlays similar to those already presented for termination loops. The overlay Cotermination-in-stream, shown in Table 9-O abstracts the stream of co-iterands seen before the exit test, similar to Termination-in-stream. Cotermination-fail-stream abstracts the stream of co-iterands seen in an environment where the test is known to have failed, similar to Termination-fail-stream. Finally, the stream of co-iterands in the steady state is abstracted by the overlay Steady-state-costream.

Given these overlays, Iterative-cosearch, as in the LENGTH program below, can be modelled as the temporal implementation of a Co-earliest operation, as shown in Table 9-P. The specifications of Co-earliest are similar to those of Earliest. Co-earliest takes as input two sequences (Input and Co-input), and returns as output the term of the Co-input sequence which corresponds to the term of the Input sequence which would be returned by Earliest.

Table 9-N. Temporal Earliest.

TemporalOverlay **temporal-earliest:** iterative-search → earliest
correspondences
iterative-search.exit.if.criterion = earliest.criterion
∧ list>sequence(steady-state-stream(iterative-search)) = earliest.input
∧ iterative-search.exit.end.output = earliest.output
∧ iterative-search.exit.end.out = earliest.out

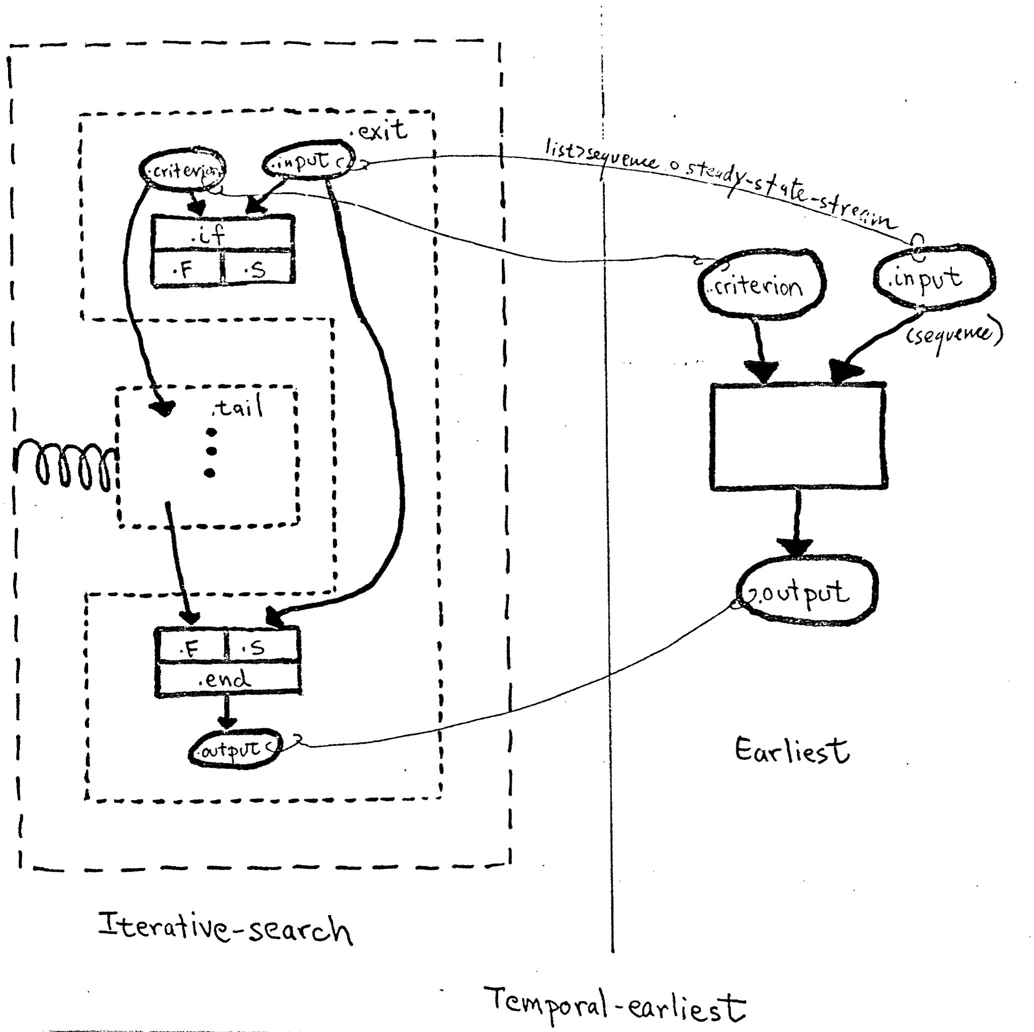


Figure 9-15. Temporal Abstraction of Search Loop.

Table 9-O. Cotermination Stream Overlays.

TemporalOverlay **cotermination-in-stream**: iterative-cotermination → finite-list
  correspondences
    iterative-cotermination.co-iterand = finite-list.head
    ∧ (*if* iterative-cotermination.tail = nil
      then nil
      else cotermination-in-stream(iterative-cotermination.tail))
        = finite-list.tail

TemporalOverlay **cotermination-fail-stream**: iterative-cotermination → finite-list+nil
  definition L = cotermination-fail-stream(T) ≡
    ∀s list>sequence(L,s) = butlast(cotermination-in-stream(T))

TemporalOverlay **steady-state-costream**: iterative-cotermination → list
  correspondences
    iterative-cotermination.co-iterand = list.head
    ∧ steady-state-costream(iterative-termination>steady-state(iterative-cotermination))
        = list.tail

Table 9-P. Temporal Co-earliest.

IOspec **co-earliest** / .input(sequence) .criterion(predicate) .co-input(sequence)
    ⇒ .output(object)
  preconditions le(length(.input),length(.co-input))
    ∧ ∃i apply(.criterion,apply(.input,i)) = true
  postconditions ∃i [apply(.co-input,i) = .output
      ∧ apply(.criterion,apply(.input,i)) = true
      ∧ ∀j (lt(j,i) ⊃ apply(.criterion,apply(.input,j)) = false)]

TemporalOverlay **temporal-co-earliest**: iterative-cosearch → co-earliest
  correspondences
    iterative-cosearch.exit.if.criterion = co-earliest.criterion
    ∧ list>sequence(steady-state-stream(iterative-cosearch)) = co-earliest.input
    ∧ list>sequence(steady-state-costream(iterative-cosearch)) = co-earliest.co-input
    ∧ iterative-cosearch.co-iterand = co-earliest.output
    ∧ iterative-cosearch.exit.end.out = co-earliest.out

```
(DEFINE LENGTH
   (LAMBDA (L)
      (PROG (N)
            (SETQ N 0)
         LP (COND ((NULL L)(RETURN N)))
            (SETQ L (CDR L))
            (SETQ N (1+ N))
            (GO LP))))
```

Thus this program can be temporally abstracted (as shown on the left of Fig. 9-16) as two instances of Iterate, one with a Cdr-iterator as input, and one with Natural-iterator as input, feeding into an instance of Co-earliest with a criterion of Null. What this figure also shows is how this plan implements @Length, the computation of the length of a sequence. If the Cdr-iterator input on the left hand side is the implementation of the spine of the sequence viewed as a labelled thread, then the output of Co-earliest is the length of the sequence.

The idea of truncating a sequence based on the occurrence of a term satisfying a given predicate in another (parallel) sequence is expressed by the Cotruncate specification defined in Table 9-Q. This specification is similar to Truncate, except that the output sequence is some initial subsequence of a second input sequence, Co-input. Furthermore, it follows from these specifications that if an instance of Truncate and of Cotruncate have the same Input sequence and Criterion, the outputs are the same length. Cotruncate is implemented temporally by Iterative-cotermination, as shown in the overlay in Table 9-Q, which resembles the Temporal-truncate overlay.

Table 9-Q. Temporal Cotruncate.

IOspec **cotruncate** / .input(sequence) .criterion(predicate) .co-input(sequence)
$\Rightarrow$ .output(finite-sequence)

properties $\forall TC$ [[instance(truncate,T) $\wedge$ instance(cotruncate,C)
$\wedge$ T.input = C.input $\wedge$ T.criterion = C.criterion $\wedge$ T.in = C.in]
$\supset$ length(sequence(T.output,T.out)) = length(sequence(C.output,C.out))]

preconditions le(length(.input),length(.co-input))
$\wedge$ $\exists i$ apply(.criterion,apply(.input,i)) = true

postconditions
$\forall i$ [index(.output,i) $\leftrightarrow$ $\forall j$ [le(j,i) $\supset$ apply(.criterion,apply(.input,j)) = false]]
$\wedge$ $\forall i$ (index(.output,i) $\supset$ apply(.output,i) = apply(.co-input,i))
$\wedge$ apply(.criterion,apply(.input,oneplus(length(.output)))) = true

TemporalOverlay **temporal-cotruncate:** iterative-cotermination $\rightarrow$ cotruncate

correspondences
iterative-cotermination.exit.if.criterion = cotruncate.criterion
$\wedge$ list>sequence(steady-state-stream(iterative-cotermination)) = cotruncate.input
$\wedge$ list>sequence(steady-state-costream(iterative-cotermination)) = cotruncate.co-input
$\wedge$ list>sequence(cotermination-fail-stream(iterative-cotermination)) = cotruncate.output
$\wedge$ iterative-cotermination.exit.end.out = truncate.out

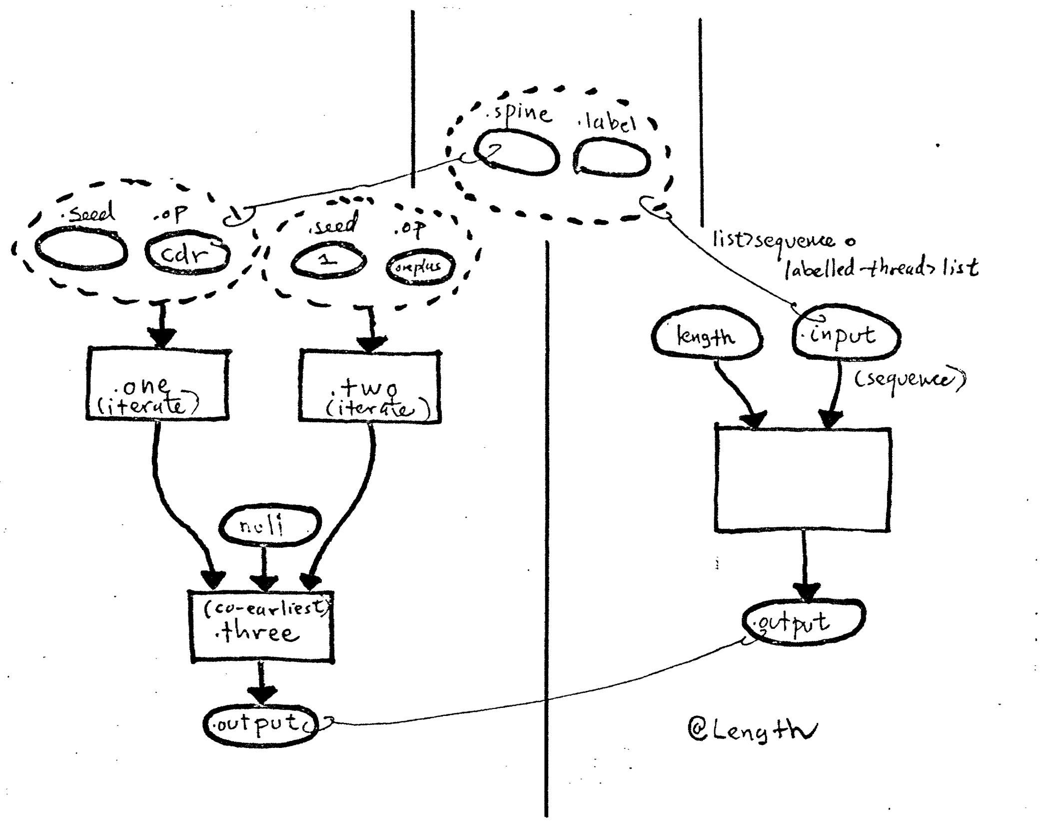


Figure 9-16. Computing the Length of a Sequence.

Table 9-R. Truncated List Generation.

TemporalPlan **truncated-list-generation** *extension* list-generation
  properties $\forall P$ [instance(truncated-list-generation,P)
    $\supset$ $\forall s$ list>sequence(truncated-generation>list(P),s) = sequence(P.three.output,P.three.out)]
  roles .one(iterate) .two(map) .three(cotruncate)
  constraints .one.output = .three.input $\wedge$ .two.output = .three.co-input
    $\wedge$ cflow(.two.out,.three.in)

TemporalOverlay **truncated-generation>list:** truncated-list-generation → finite-list
  definition L = truncated-generation>list(P) $\equiv$
    $\exists TR$ [instance(labelled-thread,T) $\wedge$ instance(terminated-thread,R)
      $\wedge$ $\forall s$ [T = list>labelled-thread(L,s)
        $\wedge$ digraph(T.spine,s) = truncated>digraph(R,s)
        $\wedge$ function(T.label,s) = function(P.two.op,P.two.in)
        $\wedge$ digraph(R.base,s) = generator>digraph(P.one.input,P.one.in)
        $\wedge$ predicate(R.criterion,s) = predicate(P.three.criterion,P.three.in)]]

TemporalPlan **car+cdr+null**
  specialization truncated-list-generation
  extension car+cdr
  properties $\forall P$ [instance(car+cdr+null,P) $\supset$
    dotted-pair>list(P.one.input.seed,P.one.in) = truncated-generation>list(P)]
  roles .one(iterate) .two(map) .three(cotruncate)
  constraints .three.criterion = null

One of the most common cliches in Lisp programming,, i.e. CDRing down a list until NULL, using the CARS, can be analyzed as the composition of an instance of Iterate, Map, and Cotruncate. The plan for this in general is called Truncated-list-generation, as shown in Table 9-R and Fig. 9-17. This plan is an extension of List-generation, discussed earlier in this section. Similar to List-generation, the final output sequence of this plan (in this case the output of role Three), viewed as a list, is the same as the labelled thread whose spine is generated by the input to the Iterate operation and truncated by the criterion of the Cotruncate operation, and whose label is the function applied in the Map operation. This relationship is expressed by the overlay Truncated-generation>list in Table 9-R, and shown in Fig. 9-17.

The specialization of Truncated-list-generation for Lisp lists in particular, where the iterator function is Cdr, the Map function is Car, and the Cotruncate predicate is Null, is called Car+cdr+null.

```
(PROG (L)
      ...
   LP (COND ((NULL L)(RETURN ...)))
      ...(CAR L)...
      (SETQ L (CDR L))
      (GO LP))
```

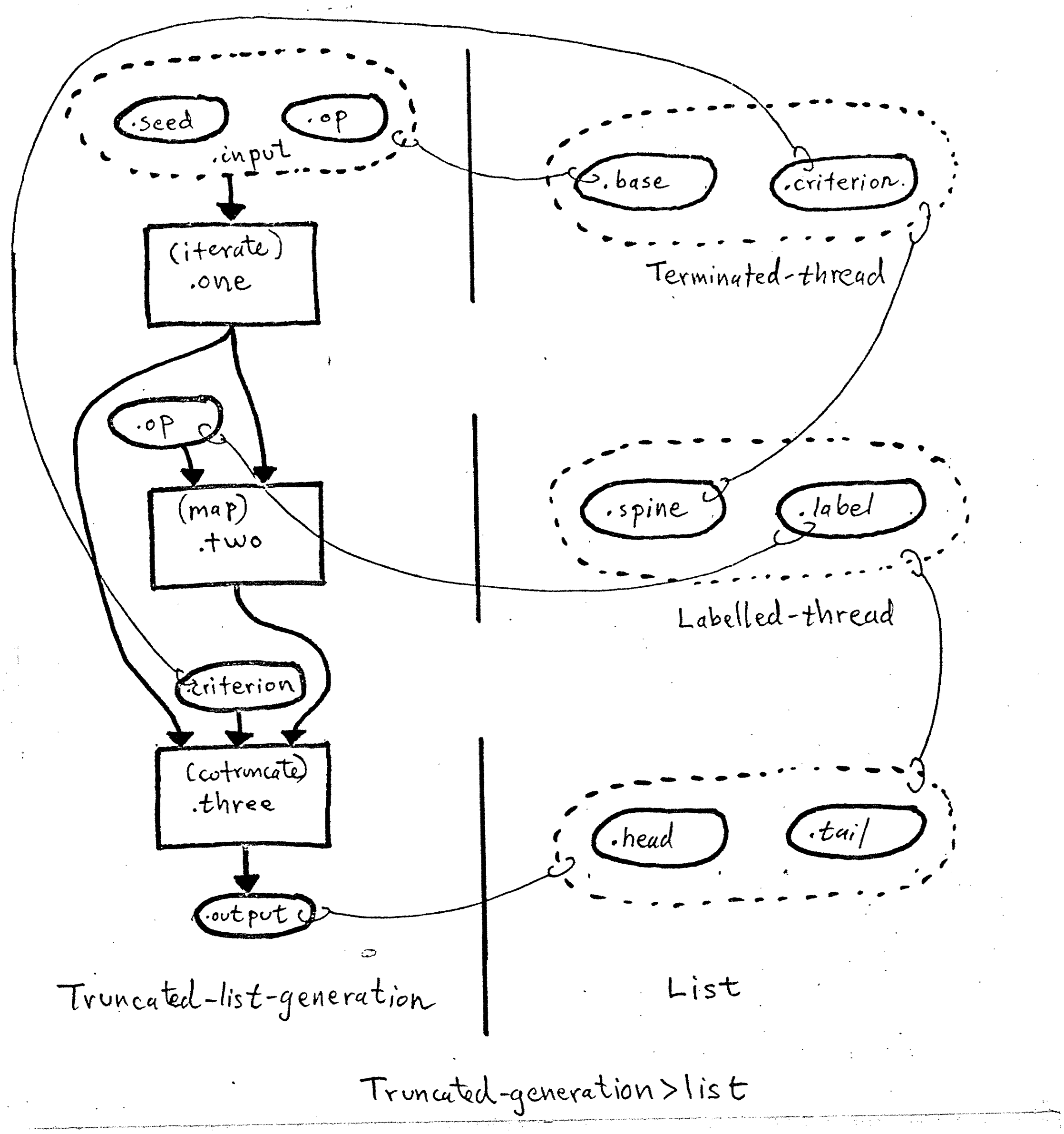


Figure 9-17. Truncated List Generation.

The output of role Three (an instance of Cotruncate) in the Car+cdr+null plan corresponds to the sequence of values returned by CAR at the underlined point in the code above. This sequence, viewed as a list, is the same as the list implemented by the dotted pair which is the seed of the iterator input to Iterate, according to the overlay Dotted-pair>list.

Temporal Sets

In many programming applications, the order in which data is generated in a loop or recursion doesn't matter. In such cases it is appropriate to take temporal abstraction one step further and talk about the *set* of objects which fill a given role in a recursive temporal plan.[1] This abstraction step is added to the existing temporal abstraction framework by using the List>set overlay.

For example, a generation loop can be thought of as the temporal implementation of a transitive closure operation, in which the binary relation being closed is many-to-one. The overlay between these two views is shown in Table 9-S and in Fig. 9-18. On the left of the overlay is a generation loop; on the right is the application of transitive closure to an iterator. The correspondence between the iterator input and the loop is the one already specified by Temporal-iterator, i.e. the Op of the Action (viewed as a relation) is the Op of the iterator, and initial input to the Action is the Seed. The output set of the transitive closure operation is the stream generated at the input of the Action (as formalized by Generation-stream), viewed as a set.

Similar temporal overlays can be constructed for other input-output specifications with sets, such as Each, Set-find, Restrict, and Any. Iterative temporal overlays for Each and Set-find are shown in Table 9-T. Iterative-temporal-each is just like Temporal-map, except rather than abstracting the streams of inputs and outputs to an iterative application as sequences, they are abstracted as sets. Iterative-temporal-find is just like Temporal-earliest, except the steady state stream of inputs is also abstracted as a set.

Table 9-S. Temporal Transitive Closure.

IOspec **@transitive-closure-iterator** / .input(iterator) ⇒ .output(set)
  postconditions .output = transitive-closure(.input,.in)

TemporalOverlay **iterative-temporal-transitive-closure:**
    iterative-generation → @transitive-closure-iterator
  correspondences
    temporal-iterator(iterative-generation) = @transitive-closure-iterator.input
    ∧ list>set(generation-stream(iterative-generation))
        = @transitive-closure-iterator.output
    ∧ iterative-generation.action.in = @transitive-closure-iterator.in

---

1. More precisely, this abstraction is appropriate when order doesn't matter and either there are no duplicates or the occurrence of duplicates doesn't matter.

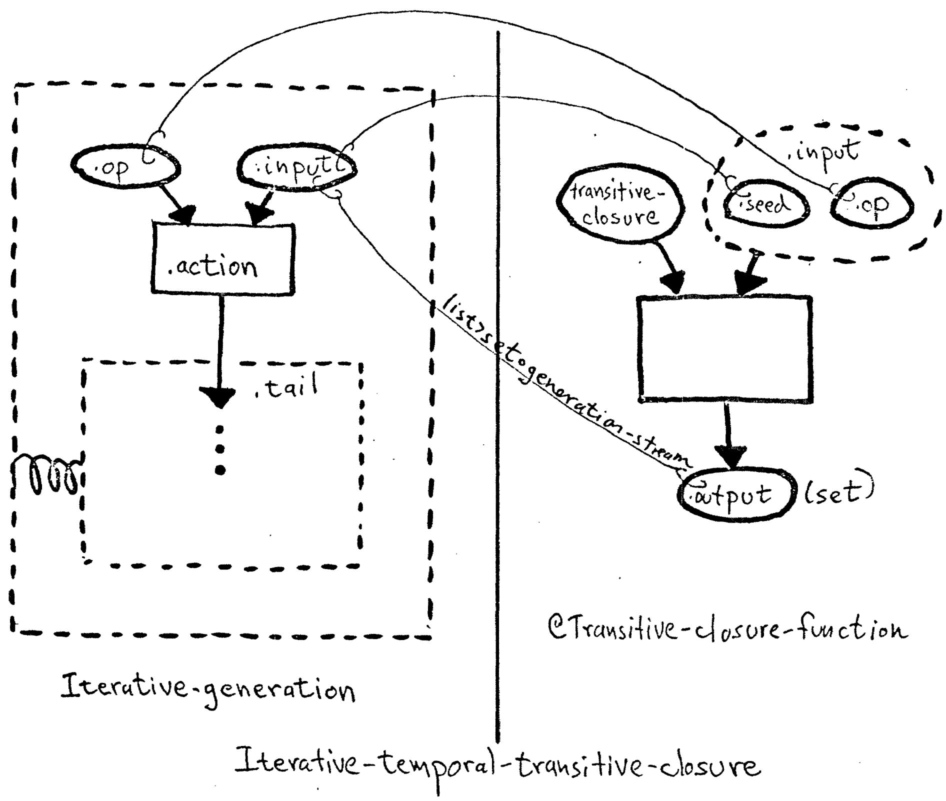


Figure 9-18. Iterative Generation as Transitive Closure.

Table 9-T. Temporal Each and Find

TemporalOverlay **iterative-temporal-each:** iterative-application → each
correspondences
iterative-application.action.op = each.op
∧ list>set(application-in-stream(iterative-application)) = each.old
∧ list>set(application-out-stream(iterative-application)) = each.new

TemporalOverlay **iterative-temporal-find:** iterative-search → set-find
correspondences
iterative-search.exit.if.criterion = set-find.criterion
∧ list>set(steady-state-stream(iterative-search)) = set-find.universe
∧ iterative-search.exit.end.output = set-find.output
∧ iterative-search.exit.end.out = set-find.out

Table 9-U. Temporal Restrict.

TemporalOverlay **filtering-in-stream:** iterative-filtering → list
correspondences iterative-filtering.filter.if.input = list.head
∧ filtering-in-stream(iterative-filtering.tail) = list.tail

TemporalOverlay **filtering-succeed-stream:** iterative-filtering → list
definition S = filtering-succeed-stream(F) ≡
[[F.filter.if.fail ≠ ⊥ ⊃ S = filtering-succeed-stream(F.tail)]
∧ [F.filter.if.succeed ≠ ⊥
⊃ [S.head = F.filter.if.input ∧ S.tail = filtering-succeed-stream(F.tail)]]]

TemporalOverlay **iterative-temporal-restrict:** iterative-filtering → restrict
correspondences
iterative-filtering.filter.if.criterion = restrict.criterion
∧ list>set(filtering-in-stream(iterative-filtering)) = restrict.old
∧ list>set(filtering-succeed-stream(iterative-filtering)) = restrict.new

---

Table 9-U shows the temporal implementation of Restrict as a filtering loop. The overlays Filtering-in-stream and Filtering-succeed-stream (see Fig. 9-19) describe how to temporally abstract the stream of input values to the test of a filtering loop and the stream of values seen in an environment where the test has succeeded (i.e. in the Then role). The definition of Filtering-in-stream has the same recursive form as the temporal overlays for iterative generation and application introduced earlier. Filtering-succeed-stream is more complicated. The basic idea of this definition is to skip the inputs which do not satisfy the test predicate. This is done by defining the stream abstraction when the test fails to be

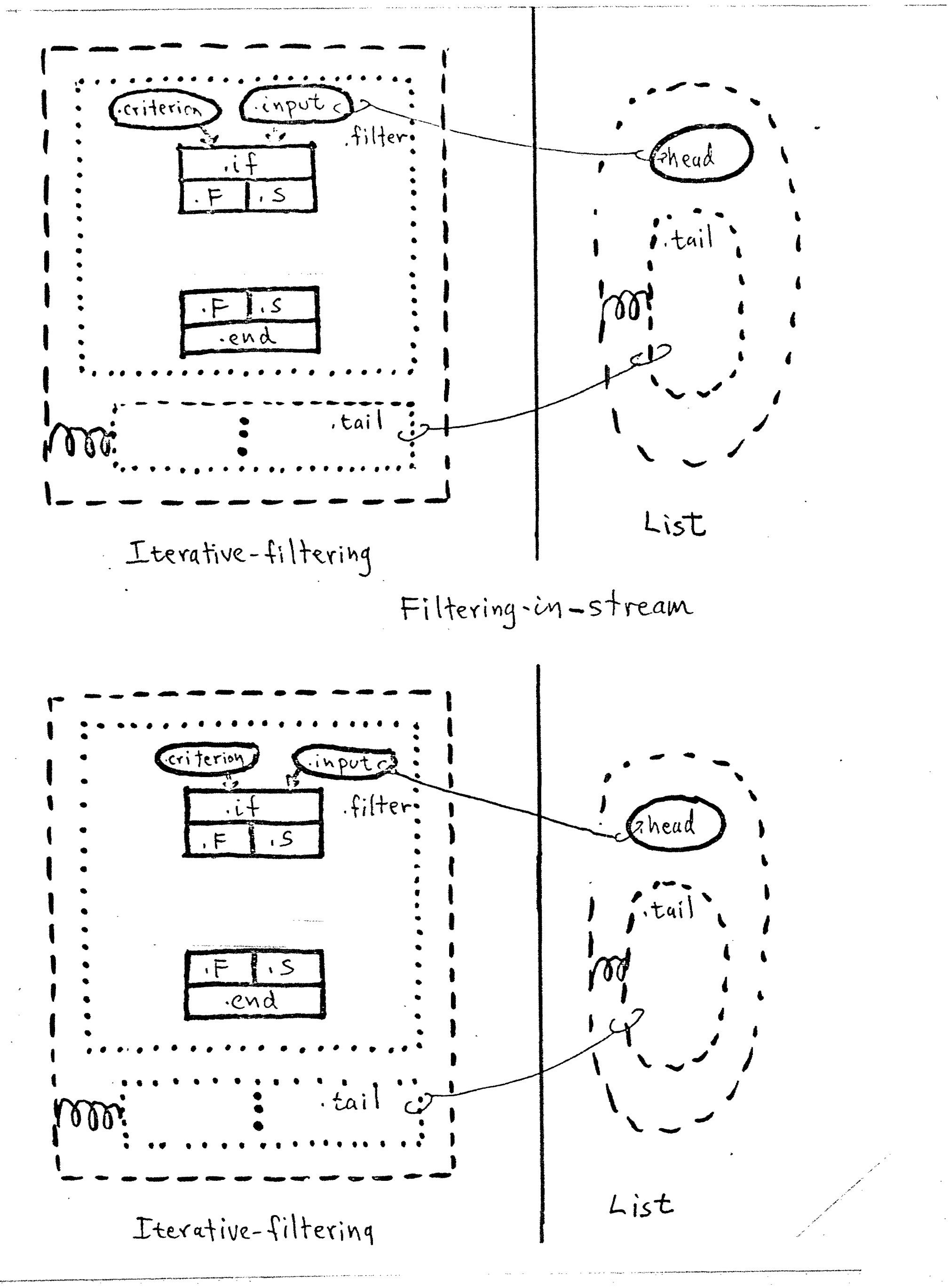


Figure 9-19. Stream Abstractions of Iterative Filtering.

the same as the stream abstraction of the next time around the loop (i.e. the tail of the recursive plan).[1]

The last overlay in Table 9-U is Iterative-temporal-restrict, also shown in Fig. 9-20. This overlay has a similar structure to all the other temporal set overlays in this section. The Old set input to Restrict corresponds to the input values of the filter test. The New set output corresponds to the input values selected by the succeed case. The Criterion of Restrict is the same as the Criterion of the filter test.

The last temporal overlay in this section is an example of how to temporally abstract a loop with two exits (a specialization of Cascade-iterative-termination). In particular, we consider here the plan Terminated-iterative-search, defined in Table 9-V and shown on the left of Fig. 9-21, in which the second exit test (If-two) is performing a search. This means that this test is an instance of @Predicate, and when the test succeeds, the input tested becomes the output object of the corresponding join (End-two), just as in Iterative-search. When the first test (If-one) succeeds, it means that the search has failed. The following is an example of how this kind of loop might be coded for searching a finite Lisp list.

Table 9-V. Temporal Any.

TemporalPlan **terminated-iterative-search**
  specialization cascade-iterative-termination
  roles .if-one(test) .if-two(@predicate) .end-one(join) .end-two(join-output)
    .tail(terminated-iterative-search)
  constraints .if-two.input=.end-two.succeed-input
    ∧ .tail.end-two.output=.end-two.fail-input

TemporalOverlay **iterative-temporal-any:** terminated-iterative-search → any
  correspondences
    list>set(termination-in-stream(cascade>iterative-termination(terminated-iterative-search))
      =any.universe
    ∧ terminated-iterative-search.if-two.criterion=any.criterion
    ∧ terminated-iterative-search.end-two.output=any.output
    ∧ terminated-iterative-search.end-one.out=any.fail
    ∧ terminated-iterative-search.end-two.out=any.succeed

1. This is a somewhat awkward construction, but I could not think of a better way of formally defining the idea of leaving out parts of the input stream. This way of modelling filtering is also motivated by considering the general case of tree recursion, where the structure of the selected inputs has to be like the structure of the input tree with chunks missing at various places in the middle.

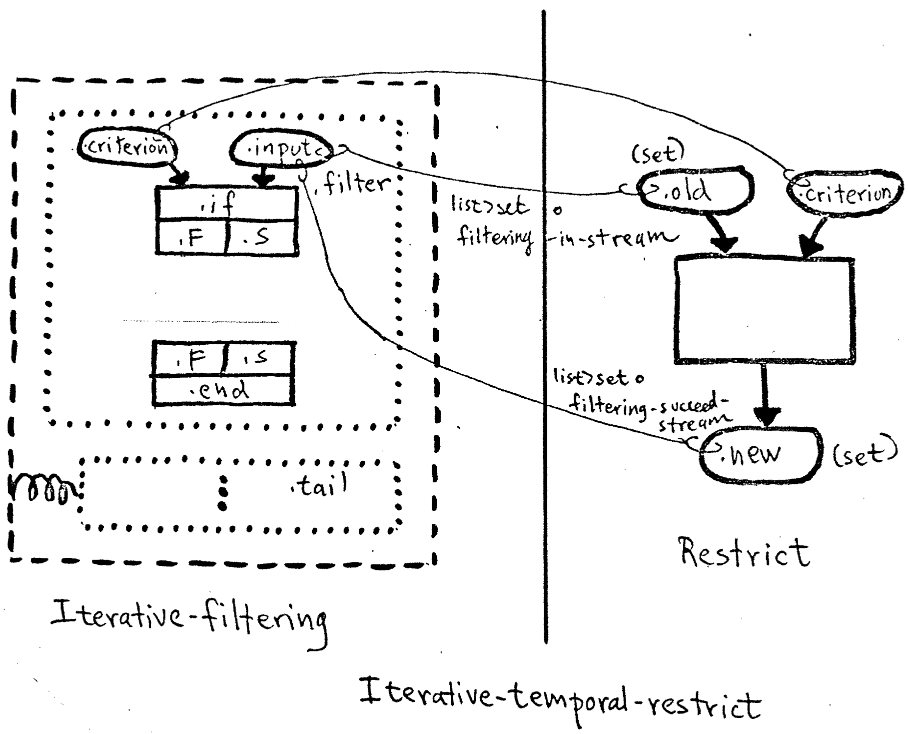


Figure 9-20. Temporal Set Abstraction of Iterative Filtering.

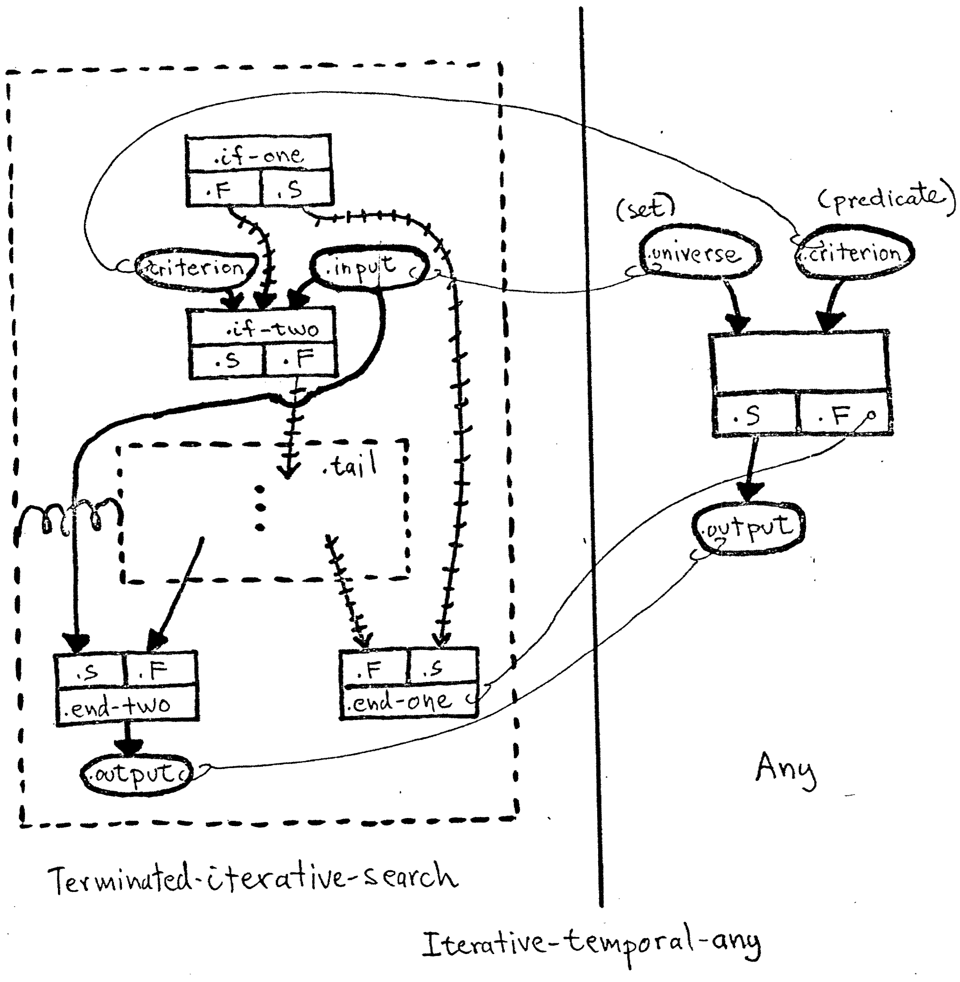


Figure 9-21. Temporal Set Abstraction of Terminated Iterative Search.

```
        (PROG (L ENTRY)
              ...
           LP (COND ((NULL L)(GO FAIL)))
              (SETQ ENTRY (CAR L))
              (COND ((P ENTRY)
                     (GO SUCCEED)))
              (SETQ L (CDR L))
              (GO LP)
      SUCCEED ...ENTRY...
              ...
         FAIL ...)
```

However a more typical way of coding Terminated-iterative-search in Lisp is illustrated by the following code from the symbol table example.

```
(DEFINE LOOKUP
   (LAMBDA (...)
      (PROG (BKT ENTRY)
            ...
         LP (COND ((NULL BKT)(RETURN NIL)))
            (SETQ ENTRY (CAR BKT))
            (COND ((...ENTRY...) (RETURN ENTRY))
            (SETQ BKT (CDR BKT))
            (GO LP)))))
```

Here rather than maintaining two control flow paths to represent the succeed and fail cases, the result of the search is encoded in a flag (ENTRY)[1] which is returned as the output of the loop. I believe this should be understood as an artifact of the restriction in Lisp that a procedure can have only one return point. The original two control flow paths are typically recovered when the procedure is invoked, as in the following code.

```
(SETQ FOUND (LOOKUP ...))
(COND (FOUND ...FOUND...)
      (T ...))
```

If the stream of inputs to the second test of Terminated-iterative-search is abstracted as a set, this plan can be viewed as the implementation of the Any specification, as shown in Table 9-V and Fig. 9-21. Specifically, the Universe of Any is the (finite) set of inputs to If-two under the assumption that that exit is never taken. The criterion of If-two corresponds to the criterion of Any; the output of End-two corresponds to the output of Any; and the output situations of End-one and End-two correspond to the Fail and Succeed situations of Any, respectively.

Accumulation

This section discusses various ways to abstract iterative accumulation programs such as the following.

1. This idea of a flag is formalized in the appendix.

```
(PROG (ACCUM ...)
      (SETQ ACCUM ...)
      ...
   LP (COND ((...)(RETURN ACCUM)))
      ...
      (SETQ ACCUM (CONS ... ACCUM))
      ...
      (GO LP))
```

To begin, the following is the temporal overlay for viewing the Input's to the Add steps of an accumulation loop as a list.

TemporalOverlay **accumulation-stream:** iterative-accumulation → list+nil
definition S = accumulation-stream(A) ≡
[[A.exit.if.succeed ≠ ⊥ ⊃ S = nil]
∧ [A.exit.if.fail ≠ ⊥ ⊃ [S.head = A.add.input ∧ S.tail = accumulation-in-stream(A.tail)]]]

This definition breaks down into two cases: if the recursion terminates on the current level, then the temporal abstraction of the inputs is Nil; otherwise, the head of the list is the current Add.Input and the tail is defined recursively.

The special case of iterative accumulation in which the Add roles are filled by instances of Push, and the Init is an instance of Nil, is called Iterative-list-accumulation, as shown in Table 9-W. Fig. 9-22 show how Iterative-list-accumulation can be viewed as the operation of making the stream of Input's to Add in the temporal viewpoint available (in reverse order) as the output of an accumulation loop.[1]

Table 9-W. List Accumulation.

TemporalPlan **iterative-list-accumulation** *specialization* iterative-accumulation
roles .exit(cond) .init(object) .add(push) .tail(iterative-list-accumulation)
constraints .init = nil

TemporalOverlay **iterative-list-accumulation>@reverse:** iterative-list-accumulation → @reverse
correspondences
list>sequence(accumulation-stream(iterative-list-accumulation)) = @reverse.input
∧ list>sequence(iterative-list-accumulation.exit.end.output) = @reverse.output
∧ iterative-list-accumulation.in = @reverse.in
∧ iterative-list-accumulation.out = @reverse.out

IOspec **@reverse** / .op(function) .input(finite-sequence) ⇒ .output(finite-sequence)
specialization @function
preconditions .op = reverse

1. Reverse is a standard relation on sequences defined in the appendix.

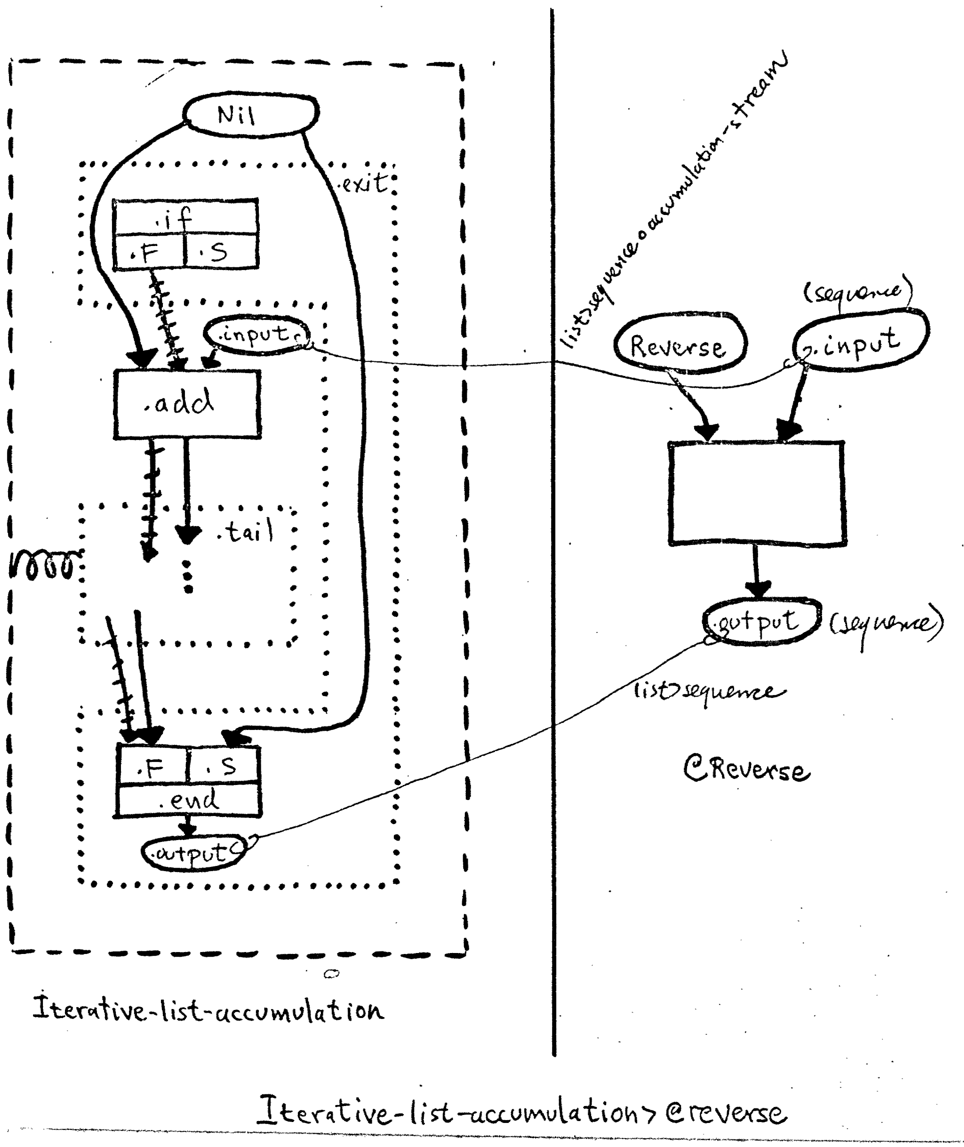


Figure 9-22. Accumulating a Stream as a List.

Thus this overlay gives a crucial correspondence between the temporal viewpoint taken inside a loop and objects, such as Exit.End.Output, which come out of the loop and are used later. For example, the following program to reverse a Lisp list[1] is analyzed as the temporal composition of an instance of Car+cdr+null, which generates the stream of inputs to CONS at the point underlined below, with an instance of Iterative-list-accumulation, which accumulates the stream in reverse order as the list in M.

```
(DEFINE REVERSE
   (LAMBDA (L)
      (PROG (M)
         LP (COND ((NULL L)(RETURN M)))
            (SETQ M (CONS (CAR L) M))
            (SETQ L (CDR L))
            (GO LP))))
```

Similarly, the special case of iterative accumulation in which the Add roles are filled by instances of Set-add, and the Init is an empty set, can be viewed as the operation of making the stream of Input's to Add in the temporal available as a set outside the loop.[2] This overlay is shown in Table 9-X and Fig. 9-23.

Another special case accumulation plan which is abstracted in terms of sets is Iterative-aggregation, shown in Table 9-Y. In this plan the Add roles are filled by applications of an aggregative function (such as Plus, Times, or Union), and the Init of the accumulation is the identity element of that function. Iterative-aggregation can be viewed as a temporal implementation of Aggregate, as defined by the overlay Temporal-aggregate in Table 9-Y and as shown in Fig. 9-24. The stream of Input's to Add, viewed as a set, corresponds to the input to Aggregate. The aggregative function applied by Add is the Binop of Aggregate. The output of the end join of the iterative plan corresponds to the output of Aggregate.

A further overlay, not shown here, can be defined to analyze accumulation loops in which the function applied is aggregative, but the Init is not the identity element; as for example a summation loop

---

Table 9-X. Set Accumulation.

TemporalPlan **iterative-set-accumulation** *specialization* iterative-accumulation
roles .exit(cond) .init(set) .add(set-add) .tail(iterative-set-accumulation)
constraints empty(.init)

TemporalOverlay **iterative-temporal-set-accumulation:** iterative-set-accumulation → set
properties $\forall A$ [instance(iterative-set-accumulation, A)
$\supset$ iterative-temporal-set-accumulation(A) = A.exit.end.output]
correspondences
list>set(accumulation-stream(iterative-set-accumulation)) = set

---

1. In which Push is implemented as CONS.
2. Notice that at this level of abstraction, we don't say exactly how this set (and therefore Set-add and Empty) are implemented.

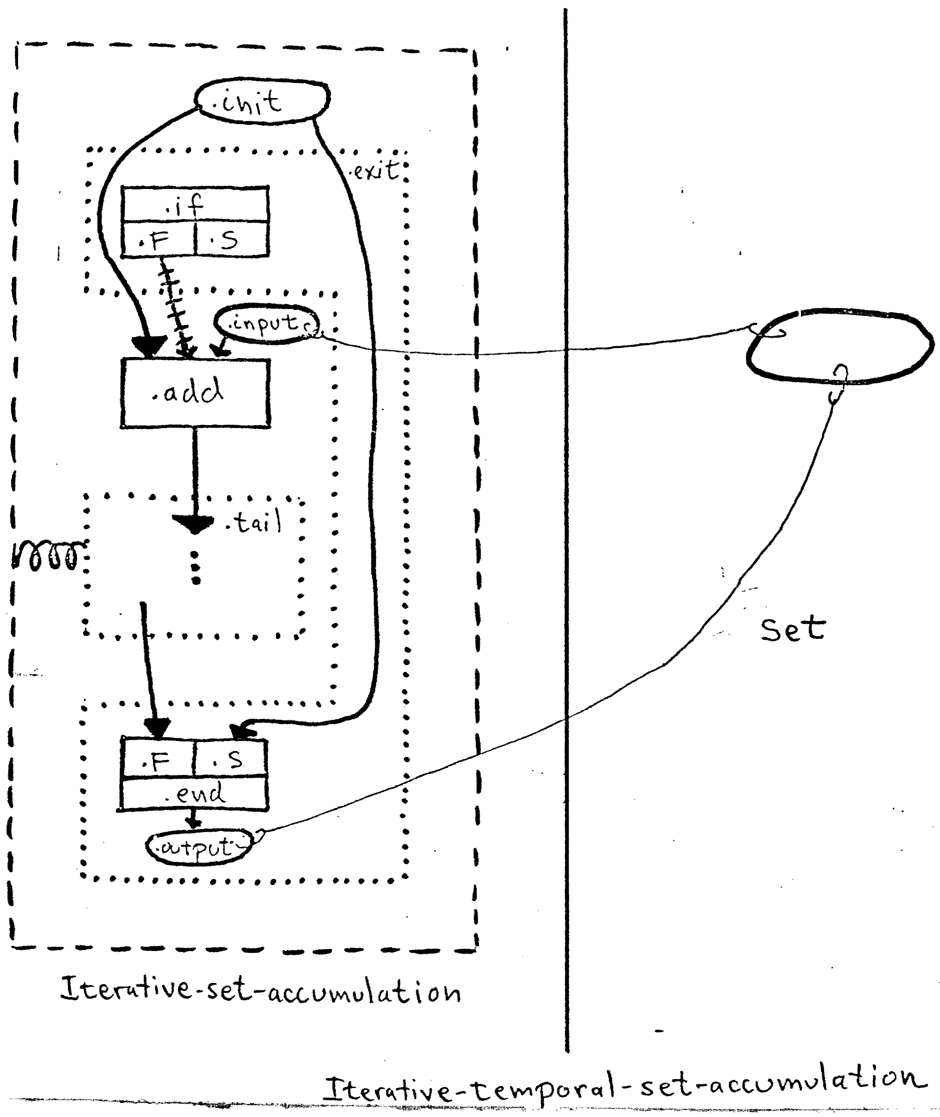


Figure 9-23. Accumulating a Temporal Set.

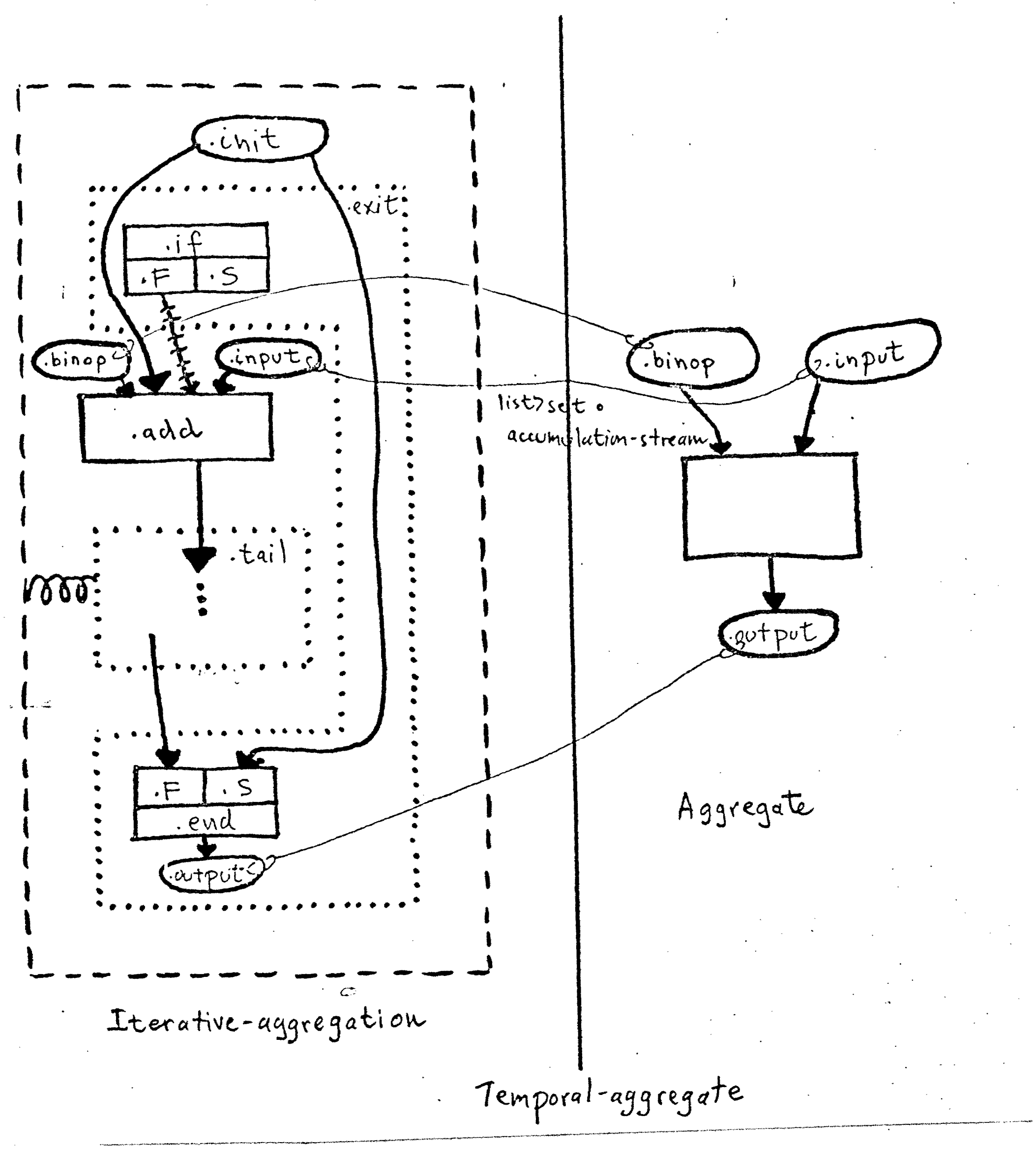


Figure 9-24. Temporally Aggregating a Set.

Table 9-Y. Temporal Aggregate.

TemporalPlan **iterative-aggregation** *specialization* iterative-accumulation
roles .exit(cond) .init(object) .add(@aggregative) .tail(iterative-aggregation)
constraints identity(.add.binop,.init)

IOspec **@aggregative** / .binop(aggregative-binfunction) .old(object) .input(object) ⇒ .new(object)
extension old+input+new
postconditions binapply(.binop,.old,.input) = .new

TemporalOverlay **temporal-aggregate**: iterative-aggregation → aggregate
correspondences
list>set(accumulation-stream(iterative-aggregation)) = aggregate.input
∧ iterative-aggregation.add.binop = aggregate.binop
∧ iterative-aggregation.exit.end.output = aggregate.output
∧ iterative-aggregation.exit.end.out = aggregate.out

---

which starts with an initial sum of 5. Such loops can be abstracted as an Aggregate operation in which the input set is obtained by adding the Init to the Accumulation-stream, viewed as a set.

Non-Iterative Temporal Abstraction

This section discusses singly recursive programs in which there is computation "on the way up", i.e. in which the recursive invocation is not the last step in the program. The kind of computation most commonly performed on the way up is accumulation, such as the following program with Lisp list accumulation on the way up.

```
(DEFINE COPYLIST
   (LAMBDA (L)
      (COND ((NULL L) NIL)
            (T (CONS (CAR L)(COPYLIST (CDR L)))))))
```

One way of thinking about this programming technique is to compare the program above with the REVERSE program of the last section, which has the same generation part, but in which the list accumulation is done iteratively ("on the way down"). This comparison is made easier by re-coding REVERSE tail recursively as shown below.[1]

1. The two different Lisp codings have the same plan.

```
(DEFINE REVERSE
   (LAMBDA (L)
      (REVERSE1 L NIL))))

(DEFINE REVERSE1
   (LAMBDA (L M)
      (COND ((NULL L) M)
            (T (REVERSE1 (CDR L)(CONS (CAR L) M))))))
```

In effect, the non-iterative program above is using the stack provided by the Lisp language implementation to reverse the order of objects flowing from the list generation to the list accumulation, in order to cancel out the order reversal introduced by the accumulation. The rest of this section will show how to formalize this way of understanding accumulation on the way up in terms of the corresponding iterative accumulation with an intervening order reversal.[1] Similar plans and overlays for other basic recursive computations on the way up (generation, application, etc.) can be constructed, but are much less common in typical programming use.[2]

In terms of the plan calculus, the difference between iterative and non-iterative singly recursive (linear) temporal plans is whether there is anything but instances of Join (or Join-output) after the recursive invocation (Tail). Instances of Join and Join-output on the way up are required in iterative plans to specify via control flow that the entire computation ends when any of its tails end, and to return any final values. In the plan for the COPYLIST program above, which is non-iterative, an instance of @Function (the Add role of the accumulation) comes after the tail. The plans for iterative and non-iterative linear accumulation can be compared diagrammatically on the right and left sides, respectively, of Fig. 9-25.

Table 9-Z defines these two plans as specializations of a more general plan called Linear-accumulation.[3] The constraints on this plan require only that the accumulation function applied (Add.Binop) be the same each time, and that the Add step occur once at each level in the recursion except when the exit test succeeds. Also, in both the iterative and non-iterative versions, the Init object is returned when the recursion terminates on the very first exit test.

Iterative-accumulation is obtained as a specialization of Linear-accumulation by adding the constraint that the Add step precedes the Tail, so that accumulation is done on the way down. There is then data flow from Add.New to Tail.Add.Old. Also, in this form of accumulation, the Init at each level is the same as the output of the preceding Add. This can be seen in the REVERSE program above, in which the value returned is `M`, which is set to `(CONS (CAR L) M)` by the preceding repetition.

---

1. The cancellation between the reversal of the order of inputs on the way up and the reversal introduced by the iterative accumulation is a particular property of List-accumulation. The reversal on the way up is a general property of non-iterative temporal abstraction.
2. In fact, even the other common accumulations, other than list accumulation, are seldom done on the way up, since the order reversal is immaterial when the streams are viewed as sets.
3. This table contains an equivalent restatement of the Iterative-accumulation plan introduced earlier, where only loops were being considered.

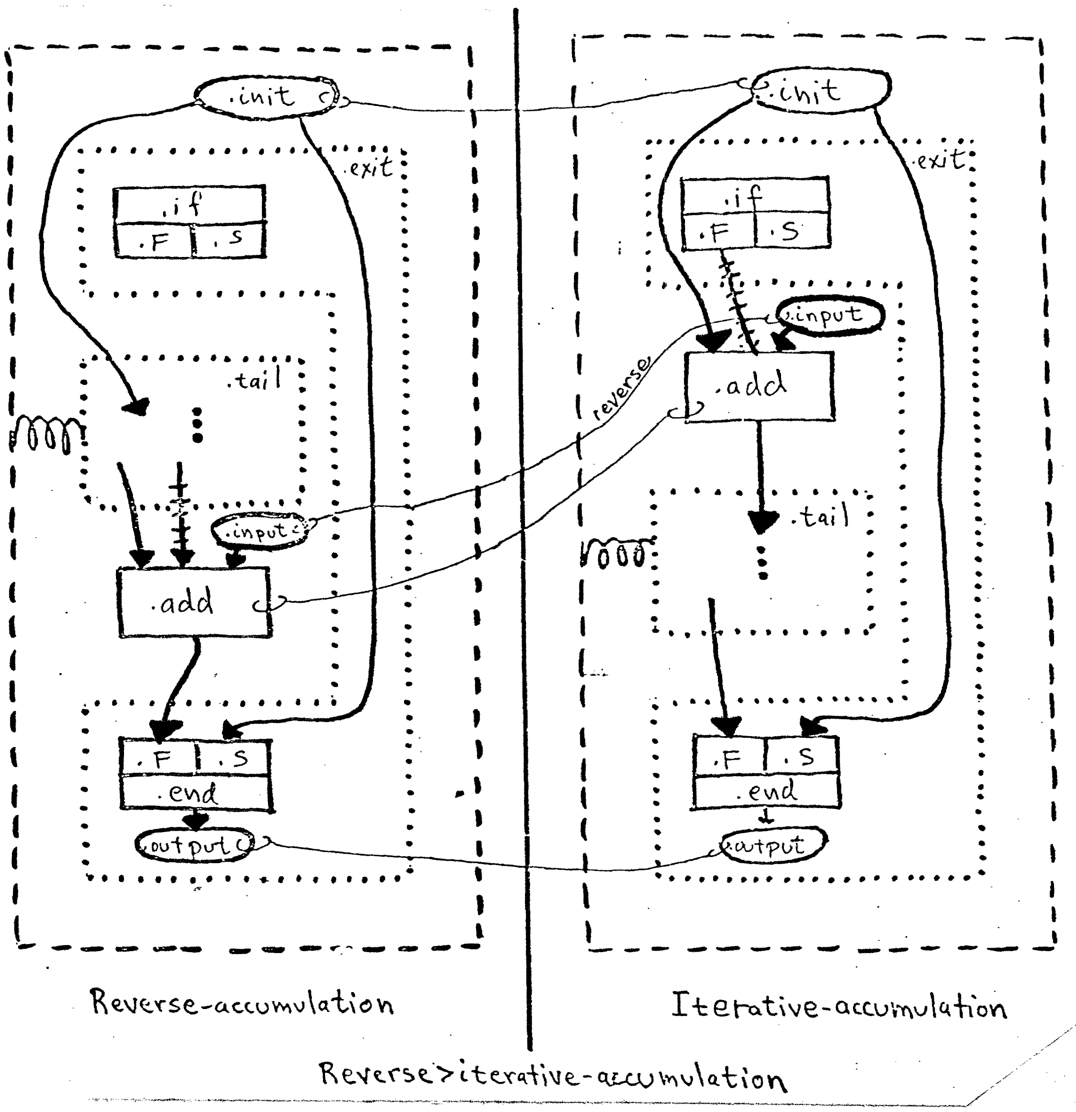


Figure 9-25. Accumulation on the Way Down vs. Up.

Table 9-Z. Iterative and Non-iterative Accumulation.

TemporalPlan **linear-accumulation** *extension* iterative-termination
roles .exit(cond) .init(object) .add(old+input+new) .tail(linear-accumulation)
constraints .init = .exit.end.succeed-input
$\wedge$ (.exit.if.fail $\neq \perp \leftrightarrow$ .add.in $\neq \perp$) $\wedge$ (.add.in $\neq \perp \leftrightarrow$ .tail.exit.if.in $\neq \perp$)

TemporalPlan **iterative-accumulation**
specialization linear-accumulation iterative-termination-output
roles .exit(cond) .init(object) .add(old+input+new) .tail(iterative-accumulation)
constraints .add.old = .init $\wedge$ .add.new = .tail.add.old $\wedge$ precedes(.add.out,.tail.add.in)

TemporalPlan **reverse-accumulation** *specialization* linear-accumulation
roles .exit(cond) .init(object) .add(old+input+new) .tail(reverse-accumulation)
constraints instance(join-output,.exit.end)
$\wedge$ .tail.exit.end.output = .add.old
$\wedge$ .add.new = .exit.end.fail-input
$\wedge$ .init = .init.tail $\wedge$ precedes(.tail.add.out,.add.in)

Reverse-accumulation, the plan for the non-iterative case, is obtained as a specialization of Linear-accumulation by constraining the Add step to follow the Tail, so that accumulation is done of the way up, and adding data flow to the Add.Old to Tail.Add.New, via the join of the Tail. Also, in this form of accumulation, the Init is the same at each level, as can be seen in the COPYLIST program above, in which NIL is returned from whichever recursive invocation finally causes the COND to succeed.

Given this framework, the Accumulation-stream overlay can be generalized to apply to instances of either Iterative-accumulation or Reverse-accumulation.

Finally, as shown in Table 9-AA and Fig. 9-25, the implicit order reversal of accumulation on the way up (as compared to on the way down) can now be modelled as an overlay, Reverse>iterative-accumulation, which establishes a correspondence between these two versions in which the type of the Add operations, the Init's, and final outputs correspond, but the accumulation input streams are reversed.

Table 9-AA. Temporal Reverse.

TemporalOverlay **reverse>iterative-accumulation:** reverse-accumulation $\rightarrow$ iterative-accumulation
correspondences
list>sequence(reverse(accumulation-stream(reverse-accumulation)))
= list>sequence(accumulation-stream(iterative-accumulation))
$\wedge$ reverse-accumulation.init = iterative-accumulation.init
$\wedge$ reverse-accumulation.add = iterative-accumulation.add
$\wedge$ reverse-accumulation.exit.end.output = iterative-accumulation.exit.end.output
$\wedge$ reverse-accumulation.exit.end.out = iterative-accumulation.exit.end.out

Similar overlays can be constructed between the iterative and non-iterative versions of other recursive plans, such as generation, application, etc.

9.4 Recursive Structures

This section sketches how the epistemology of singly recursive data structures (lists, etc.) and temporal plans (loops, etc.) of the last two sections can be generalized to double and multiple recursion. Only a small amount of formal definition will be presented in this section, however, since the plans and overlays for double and multiple recursive structures tend to be longer and more detailed than those for linear structures, without introducing any fundamentally new ideas.

Table 9-AB shows the basic idea of double recursion. The data plan Double-recursion has two roles, Left and Right, which are either themselves instances of Double-recursion, or of type Atom, which is a primitive type used to terminate multiple recursions. Finite double recursion is defined analogously to finite single recursion. Recursion with a varying number of recursive instances at each level can be defined in terms of a single role which is constrained to be a set, each of which is either a recursive instance or an atom.

The doubly recursive data structure analogous to List is Binlist (binary list), a data structure with one head and two "tails", Left and Right. The binary data structure corresponding to Thread in the linear case is Bintree, and in the general case, Tree. In Lisp programming, binary trees are a more common data structure than binary lists, since a binary tree may be easily constructed out of dotted pairs, as described by Car-cdr-generator (see Table 9-AC). A double recursion may be viewed as a binary tree in which the left and right recursive instances correspond to subtrees whose roots are successors of the root of the binary tree, as specified by the following overlay.

Table 9-AB. Double Recursion.

DataPlan **double-recursion**
  roles .left(double-recursion+atom) .right(double-recursion+atom)

Type **atom**

Type **double-recursion+atom** *uniontype* double-recursion atom

DataPlan **binlist** *extension* double-recursion
  roles .head(object) .left(binlist+atom) .right(binlist+atom)

Type **binlist+atom** *uniontype* binlist atom

DataOverlay **double-recursion>bintree:** double-recursion+atom → bintree
definition T=double-recursion>bintree(R,s) ≡
[[instance(atom,double-recursion+atom(R,s)) ↔ terminal(T,root(T))]
∧ [instance(double-recursion,double-recursion+atom(R,s)) ⊃
∀xy [root(T,x) ∧
[root(double-recursion>bintree(double-recursion+atom(R,s).left,s),y)
∨ root(double-recursion>bintree(double-recursion+atom(R,s).right,s))]
⊃ successor(T,x,y)]]]

The basic plans for unbounded iterative computation introduced earlier in this chapter, generation, application, termination, filtering, and accumulation, can also be generalized to double and multiple recursion. For example, Table 9-AC and Fig. 9-26 show the plan for doubly recursive generation, such as in the following code.

```
(DEFINE GENERATE
   (LAMBDA (S)
      ...(GENERATE (CAR S))...
      ...(GENERATE (CDR S))...))
```

The overlay Temporal-binary-generator is analogous to Temporal-iterator for loops. It specifies how Binary-generation can be viewed as the temporal implementation of the generator for a binary tree. For example, this is the overlay which relates the code above to the Lisp binary tree generator Car-cdr-generator.

Table 9-AC. Binary Generation.

TemporalPlan **binary-generation** *extension* double-recursion
roles .current(object) .action-left(@function) .action-right(@function)
.left(binary-generation) .right(binary-generation)
constraints .current=.action-left.input ∧ .current=.action-right.input
∧ .left.action-left.op=.action-left.op ∧ .right.action-right.op=action-right.op
∧ .action-left.output=.left.current
∧ .action-right.output=.right.current

TemporalOverlay **temporal-binary-generator:** binary-generation → binary-generator
correspondences binary-generation.current=binary-generator.seed
∧ binary-generation.action-left.op=binary-generator.left
∧ binary-generation.action-right.op=binary-generator.right

DataPlan **binary-generator**
roles .seed(object) .left(function) .right(function)

DataPlan **car-cdr-generator** *specialization* binary-generator
roles .seed(dotted-pair) .left(function) .right(function)
constraints .left=car ∧ .right=cdr

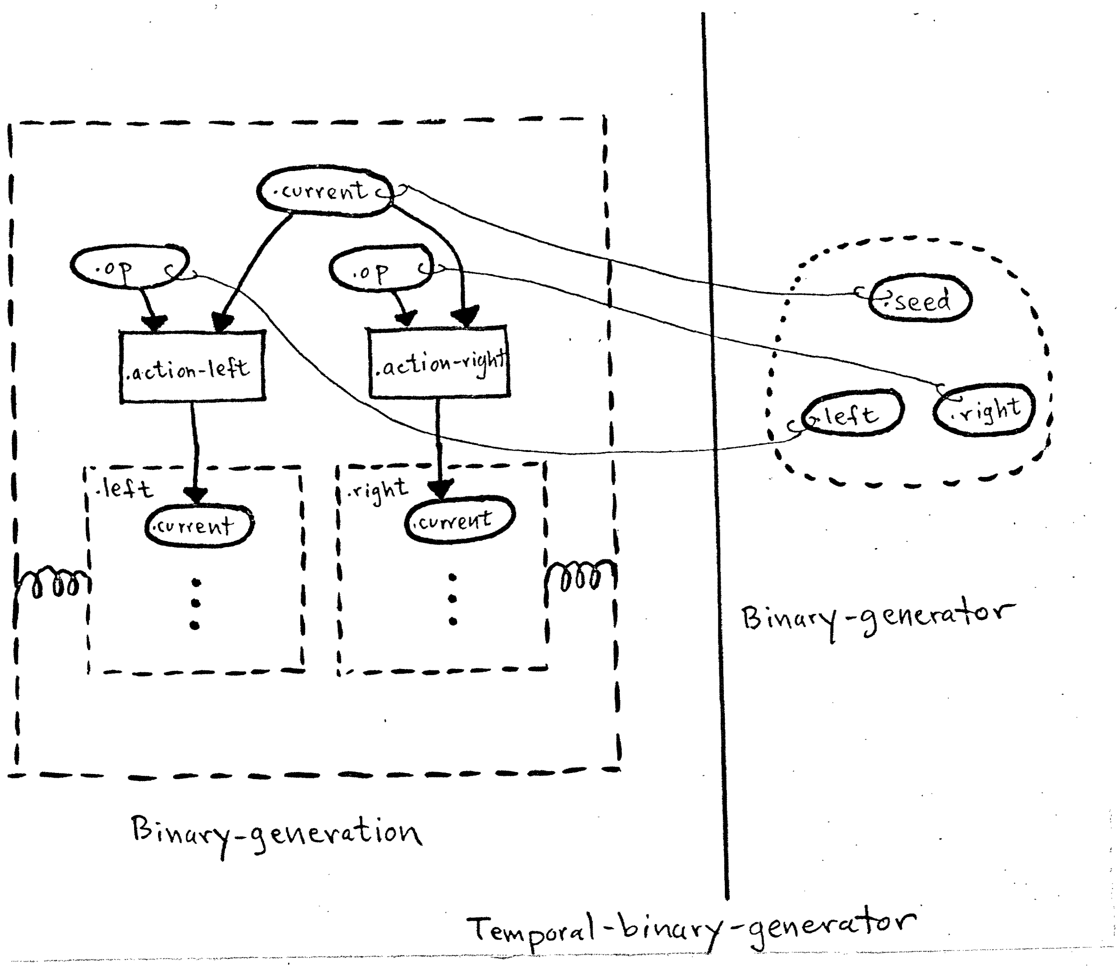


Figure 9-26. Binary Generation.

Notice that there are no constraints in the Binary-generation plan between the order of execution of the left and right recursive invocations. Standard traversal orders for binary trees (such as pre- and post-order) are represented as specializations of Binary-generation. In the temporal view, these traversal orders can be viewed as overlays which "flatten" a tree into a linear structure in different ways.

Temporal abstraction of multiple recursive plans gives rise to tree structured streams and reverse streams. Operations on these temporal abstractions are the generalizations of the corresponding operations on temporal sequences, such as Iterate, Map, Truncate, and so on. A particularly important doubly recursive temporal plan is binary tree accumulation, as in the following code.

```
(DEFINE COPYTREE
   (LAMBDA (S)
      (COND ((ATOM S) S)
            (T (CONS (COPYTREE (CAR S))
                     (COPYTREE (CDR S)))))))
```

The plan for this program is the temporal composition of binary generation with binary truncation (on ATOM), and binary accumulation in which the accumulation function constructs an instance Double-recursion from a given left and right. For binary trees in Lisp, this construction operation is implemented by CONS.

APPENDIX

PLAN LIBRARY REFERENCE

1. SETS

Type **set**

Function **size**: set $\rightarrow$ cardinal

Function **set-type**: set $\rightarrow$ type
  properties $\forall S P\,[\,\text{set-type}(S) = P \supset [\,\forall x\,[\,(x \in S) \supset \text{instance}(P,x)\,]\,]\,]$

Type **finite-set** *subtype* set
  definition $\text{instance}(\text{finite-set},S) \equiv [\,\text{instance}(\text{set},S) \wedge \text{finite}(\text{size}(S))\,]$

1.1 Relations on Sets

Predicate **empty**: set $\rightarrow$ boolean
  properties $\forall S\,[\,\text{empty}(S) \leftrightarrow \text{size}(S) = 0\,]$
  definition $\text{empty}(S) \equiv \forall x\,(x \notin S)$

Predicate **universal**: set $\rightarrow$ boolean
  definition $\text{universal}(S) \equiv \forall x\,(x \in S)$

Binrel **disjoint**: set $\times$ set $\rightarrow$ boolean
  definition $\text{disjoint}(S,T) \equiv \forall x\,\neg[\,(x \in S) \wedge (x \in T)\,]$

Binrel **subset**: set $\times$ set $\rightarrow$ boolean
  properties instance(partial-order,subset)
  definition $\text{subset}(S,T) \equiv \forall x\,[\,(x \in S) \supset (x \in T)\,]$

1.2 Input-Output and Test Specifications with Sets

IOspec **set-find** / .universe(set) .criterion(predicate) $\Rightarrow$ .output(object)
  preconditions set-type(.universe) = domain-type(.criterion)
    $\wedge\ \exists x\,[\,(x \subset \text{.universe}) \wedge \text{apply}(\text{.criterion},x) = \text{true}\,]$
  postconditions $(\text{.output} \in \text{.universe}) \wedge \text{apply}(\text{.criterion},\text{.output}) = \text{true}$

IOspec **each** / .old(set) .op(function) $\Rightarrow$ .new(set)
preconditions set-type(.old) = domain-type(.op) $\wedge$ subset(.old,domain(.op))
postconditions set-type(.new) = range-type(.op)
$\wedge\ \forall y\,[\,(y \in \text{.new}) \leftrightarrow \exists x\,[\,(x \in \text{.old}) \wedge \text{apply(.op,}x) = y\,]\,]$

IOspec **restrict** / .old(set) .criterion(predicate) $\Rightarrow$ .new(set)
preconditions set-type(.old) = domain-type(.criterion)
postconditions set-type(.new) = set-type(.old)
$\wedge\ \forall x\,[\,(x \in \text{.new}) \leftrightarrow [\,(x \in \text{.old}) \wedge \text{apply(.criterion,}x) = \text{true}\,]\,]$

IOspec **set-add** / .old(set) .input(object) $\Rightarrow$ .new(set)
specialization old+input+new-set
preconditions instance(set-type(.old),.input)
postconditions set-type(.new) = set-type(.old) $\wedge$ (.input $\in$ .new)
$\wedge\ \forall x\,[\,x \neq \text{.input} \supset [\,(x \in \text{.old}) \leftrightarrow (x \in \text{.new})\,]\,]$

IOspec **set-remove** / .old(set) .input(object) $\Rightarrow$ .new(set)
specialization old+input+new-set
preconditions instance(set-type(.old),.input)
postconditions set-type(.new) = set-type(.old) $\wedge$ (.input $\notin$ .new)
$\wedge\ \forall x\,[\,x \neq \text{.input} \supset [\,(x \in \text{.old}) \leftrightarrow (x \in \text{.new})\,]\,]$

Test **any** / .universe(finite-set) .criterion(predicate) $\Rightarrow$ .output(object)$_{\text{.succeed}}$
condition $\exists x\,[\,(x \in \text{.universe}) \wedge \text{apply(.criterion,}x) = \text{true}\,]$
postconditions (.output $\in$ .universe) $\wedge$ apply(.criterion,.output) = true

Test **member?** / .universe(set) .input(object)
condition .input $\in$ .universe

1.3 Aggregating a Set

IOspec **aggregate** / .input(finite-set) .binop(aggregative-binfunction) $\Rightarrow$ .output(object)
preconditions $\neg$empty(.input) $\wedge$ subset(.input,argtype-one(.binop).instances)
postconditions $\exists QT\,[$ instance(irredundant-sequence,Q)
$\wedge$ instance(irredundant-sequence,T)
$\wedge$ .input = sequence>set(Q,s)
$\wedge$ length(T) = length(Q)
$\wedge$ first(T) = first(Q)
$\wedge\ \forall i\,[$ index(T,i) $\wedge$ $i \neq 1 \supset$
apply(T,i) = binapply(.binop,apply(Q,i),apply(T,oneminus(i)))$\,]$
$\wedge$ .output = last(T) $]$

IOspec **sum** / .input(finite-set) .binop(aggregative-binfunction) $\Rightarrow$ .output(object)
specialization aggregate
preconditions .binop = plus

IOspec **product** / .input(finite-set) .binop(aggregative-binfunction) $\Rightarrow$ .output(object)
specialization aggregate
preconditions .binop = times

IOspec **aggregate-union** / .input(finite-set) .binop(aggregative-binfunction) $\Rightarrow$ .output(object)
specialization aggregate
preconditions .binop = union

IOspec **aggregate-intersection** / .input(finite-set) .binop(aggregative-binfunction) $\Rightarrow$ .output(object)
specialization aggregate
preconditions .binop = intersection

IOspec **max** / .input(finite-set) .binop(aggregative-binfunction) $\Rightarrow$ .output(object)
specialization aggregate
preconditions .binop = greater

IOspec **min** / .input(finite-set) .binop(aggregative-binfunction) $\Rightarrow$ .output(object)
specialization aggregate
preconditions .binop = lesser

1.4 Linear Implementations of Sets

DataOverlay **list>set**: list+nil $\rightarrow$ set
properties $\forall Ls$ [list+nil(L,s) = nil $\leftrightarrow$ empty(list>set(L,s))]
definition T = list>set(L,s) $\equiv$ $\forall x$ [($x \in T$) $\leftrightarrow$ [x = list(L,s).head $\vee$ ($x \in$ list>set(list(L,s).tail,s))]]

DataOverlay **sequence>set**: sequence $\rightarrow$ set
properties $\forall Qs$ [list>set(sequence>list(Q,s)) = sequence>set(Q,s)
$\wedge$ [instance(irredundant-sequence,sequence(Q,s)) $\supset$
length(sequence(Q,s)) = size(sequence>set(Q,s))]]
definition T = sequence>set(Q,s) $\equiv$ $\forall x$ [($x \in T$) $\leftrightarrow$ $\exists i$ apply(sequence(Q,s),i) = x]

DataOverlay **labelled-thread>set**: labelled-thread $\rightarrow$ set
properties $\forall Ls$ list>set(L,s) = labelled-thread>set(list>labelled-thread(L,s),s)
definition E = labelled-thread>set(T,s) $\equiv$
$\forall x$ [($x \in E$) $\leftrightarrow$ $\exists y$ [node(labelled-thread(T,s).base,y) $\wedge$ apply(labelled-thread(T,s).label,y) = x]]

1.5 Implementations of Set Add[1]

TemporalPlan **internal-labelled-thread-add**
roles .old(labelled-thread) .add(internal-thread-add) .update(newarg)
.new(labelled-thread)
constraints .old.spine = .add.old ∧ .old.label = .update.old
∧ .add.input = .update.arg
∧ .add.new = .new.spine ∧ .update.new = .new.label

TemporalOverlay **internal-thread>set-add:** internal-labelled-thread-add → set-add
properties ∀*P* [instance(internal-labelled-thread-add,*P*) ⊃
[instance(#internal-thread-add,*P*.add) ↔ instance(#set-add,internal-thread>set-add(*P*))]]
correspondences
labelled-thread>set(internal-labelled-thread-add.old) = set-add.old
∧ internal-labelled-thread-add.update.input = set-add.input
∧ labelled-thread>set(internal-labelled-thread-add.new) = set-add.new
∧ internal-labelled-thread.add.in = set-add.in
∧ internal-labelled-thread.update.out = set-add.out

TemporalOverlay **push>set-add:** push → set-add
properties ∀*P* [instance(push,*P*) ∧ instance(irredundant-list,*P*.output)
⊃ instance(set-add-one,push>set-add(*P*))]
correspondences list>set(push.old) = set-add.old
∧ push.input = set-add.input
∧ list>set(push.new) = set-add.new
∧ push.in = set-add.in
∧ push.out = set-add.out

1.6 Set Removal for Irredundant Lists

TemporalOverlay **@tail+internal>restrict:** @tail+internal → restrict-one
correspondences
list>set(@tail+internal.action.input) = restrict-one.old
∧ complement(@tail+internal.update.if.criterion) = restrict-one.criterion
∧ list>set(@tail+internal.update.end.output) = restrict-one.new
∧ @tail+internal.action.in = restrict-one.in
∧ @tail+internal.update.end.out = restrict-one.out

---

1. See Fig. A-1.

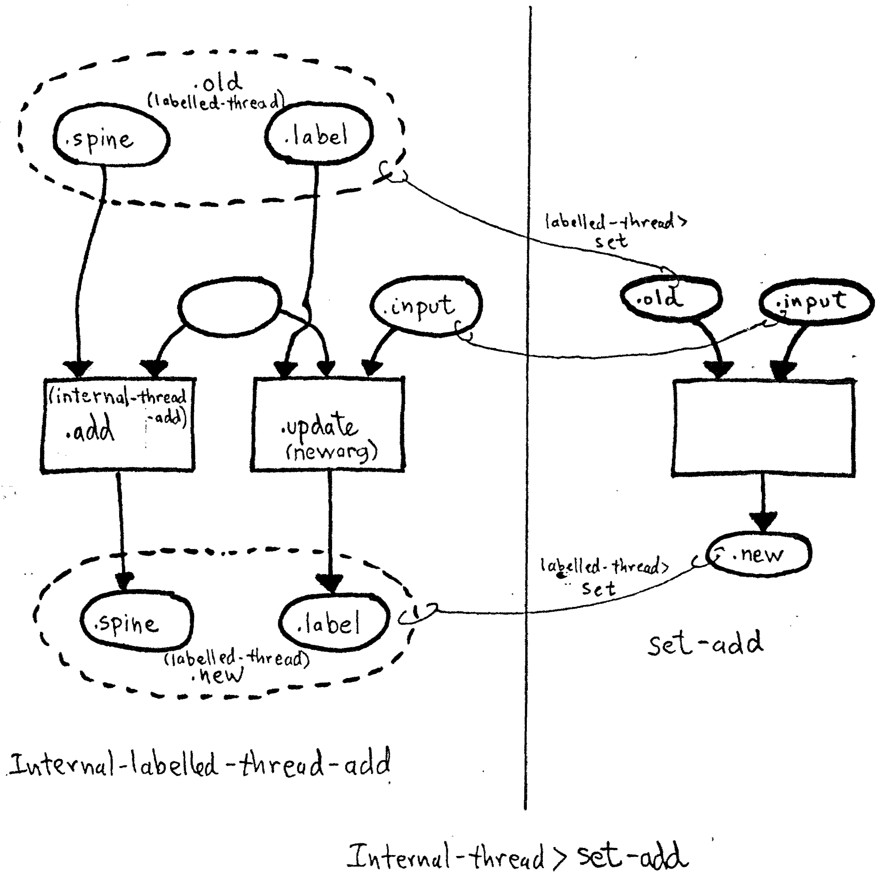


Figure A-1. Adding to a Set Implemented as a Thread.

IOspec **restrict-one** / .old(set) .criterion(predicate) $\Rightarrow$ .new(set)
specialization restrict
preconditions $\exists x$ [($x \in$ .old) $\wedge$ apply(.criterion,x) = false]
$\wedge$ $\forall xy$ [($x \in$ .old) $\wedge$ ($y \in$ .old)
$\wedge$ apply(.criterion,x) = false $\wedge$ apply(.criterion,y) = false $\supset$ $x = y$]

TemporalPlan **@tail+internal**
roles .action(@head) .update(cond)
.internal(internal-labelled-thread-find+remove)
constraints instance(@predicate,.update.if)
$\wedge$ instance(@tail,.update.then)
$\wedge$ instance(join-output,.update.end)
$\wedge$ instance(irredundant-list,.action.input)
$\wedge$ .action.output = .update.if.input $\wedge$ cflow(.action.out,.update.if.in)
$\wedge$ .action.input = .update.then.input
$\wedge$ .update.then.output = .update.end.succeed-input
$\wedge$ update.else.in = .internal.find.in $\wedge$ update.else.out = .internal.remove.out
$\wedge$ list>labelled-thread(.action.input) = .internal.old
$\wedge$ .update.if.criterion = .internal.composite.criterion
$\wedge$ list>labelled-thread(.internal.new) = .update.end.fail-input

TemporalPlan **internal-labelled-thread-find+remove** *extension* internal-thread-find+remove
roles .old(labelled-thread) .new(labelled-thread) .composite(function+predicate)
.find(internal-thread-find) .remove(internal-thread-remove)
constraints .old.spine = .find.universe $\wedge$ .new.spine = .remove.new
$\wedge$ .old.label = .new.label $\wedge$ .old.label = .composite.op
$\wedge$ function+predicate>predicate(.composite) = .find.criterion

TemporalPlan **internal-thread-find+remove**
roles .find(internal-thread-find) .remove(internal-thread-remove)
constraints .find.universe = .remove.old $\wedge$ .find.output = .remove.input

1.7 Discrimination

Type **discrimination** *subtype* function
definition instance(discrimination,F) $\equiv$ [instance(function,F)
$\wedge$ $\forall x$ [instance(domain-type(F),x) $\supset$ ($x \in$ domain(F))]
$\wedge$ $\forall bs$ [($b \in$ range(F)) $\supset$ set-type(set(b,s)) = domain-type(F)]]

DataOverlay **discrimination>set:** discrimination $\rightarrow$ set
definition Q = discrimination>set(F,s) $\equiv$ [domain-type(function(F,s)) = set-type(Q)
$\wedge$ $\forall x$ [($x \in Q$) $\leftrightarrow$ ($x \in$ set(apply(function(F,s),x),s))]]

1.8 Testing Membership in a Discrimination[1]

TemporalOverlay **discriminate+member?>member?:** discriminate+member? → member?
correspondences
discrimination>set(discriminate+member?.discriminate.op) = member?.universe
∧ discriminate+member?.discriminate.input = member?.input
∧ discriminate+member?.if.in = member?.in
∧ discriminate+member?.if.succeed = member?.succeed
∧ discriminate+member?.if.fail = member?.fail

TemporalPlan **discriminate+member?**
roles .discriminate(@function) .if(member?)
constraints instance(discrimination,.discriminate.op)
∧ .discriminate.output = .if.universe ∧ .discriminate.input = .if.input
∧ cflow(.discriminate.out,.if.in)

1.9 Updating a Discrimination

TemporalOverlay **discriminate+action+update>action:**
discriminate+action+update → old+input+new-set
properties ∀*D A* [*A* = discriminate+action+update>action(*D*) ⊃
[instance(set-add,*D*.action) ↔ instance(set-add,*A*)]
∧ [instance(set-add-one,*D*.action) ↔ instance(set-add-one,*A*)]
∧ [instance(set-remove,*D*.action) ↔ instance(set-remove,*A*)]
∧ [instance(#newvalue,*D*.update) ↔ instance(#old+input+new,*A*)]]
correspondences
discrimination>set(discriminate+action+update.discriminate.op)
= old+input+new-set.old
∧ discriminate+action+update.discriminate.input = old+input+new-set.input
∧ discriminate+action+update.action.input = old+input+new-set.input
∧ discrimination>set(discriminate+action+update.update.new)
= old+input+new-set.new
∧ discriminate+action+update.discriminate.in = old+input+new-set.in
∧ discriminate+action+update.update.out = old+input+new-set.out

1. See Fig. A-2.

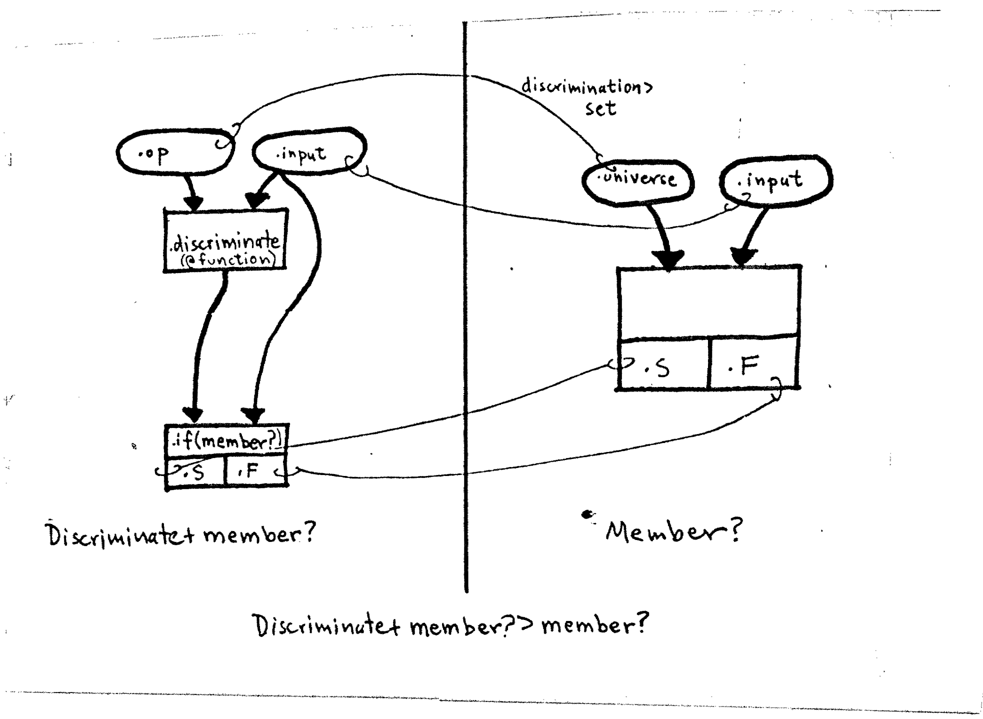


Figure A-2. Testing Membership in a Discrimination.

TemporalPlan **discriminate+action+update** *extension* action+update
roles .discriminate(@function) .action(old+input+new-set) .update(newvalue)
constraints instance(discrimination,.discriminate.op)
$\wedge$ .discriminate.output = .action.old
$\wedge$ .discriminate.output = .update.value
$\wedge$ .action.new = .update.input
$\wedge$ .discriminate.op = .update.old
$\wedge$ cflow(.discriminate.out,.action.in) $\wedge$ cflow(.action.out,.update.in)

IOspec **old+input+new-set** / .old(set) .input(object) $\Rightarrow$ .new(set)
specialization old+input+new

2. ASSOCIATIVE RETRIEVAL AND DELETION

Test **retrieve** / .universe(finite-set) .key(function) .input(object) $\Rightarrow$ .output(object)$_{\text{.succeed}}$
condition $\exists x$ [($x \in$ .universe) $\wedge$ apply(.key,x) = .input]
postconditions (.output $\in$ .universe) $\wedge$ apply(.key,.output) = .input

IOspec **expunge** / .old(finite-set) .key(function) .input(object) $\Rightarrow$ .new(finite-set)
extension old+input+new-set
postconditions $\forall x$ [($x \in$ .new) $\leftrightarrow$ [($x \in$ .old) $\wedge$ apply(.key,x) $\neq$ .input]]

IOspec **expunge-one** / .old(finite-set) .key(function) .input(object) $\Rightarrow$ .new(finite-set)
specialization expunge
preconditions $\exists x$ [($x \in$ .old) $\wedge$ apply(.key,x) = .input]
$\wedge$ $\forall xy$ [($x \in$ .old) $\wedge$ ($y \in$ .old)
$\wedge$ apply(.key,x) = .input $\wedge$ apply(.key,y) = .input $\supset$ $x = y$]

IOspec **#expunge** / .old(finite-set) .key(function) .input(object) $\Rightarrow$ .new(finite-set)
specialization expunge
postconditions .old = .new

2.1 Implementation of Associative Retrieval

TemporalOverlay **any>retrieve**: any-composite $\rightarrow$ retrieve
correspondences any-composite.universe = retrieve.universe
$\wedge$ any-composite.composite.op = retrieve.key
$\wedge$ any-composite.composite.two = retrieve.input
$\wedge$ any-composite.if.output = retrieve.output
$\wedge$ any-composite.if.in = retrieve.in
$\wedge$ any-composite.if.succeed = retrieve.succeed
$\wedge$ any-composite.if.fail = retrieve.fail

TemporalPlan **any-composite**
  roles .composite(function+two) .if(any)
  constraints function+two>predicate(.composite) = .if.criterion

2.2 Implementation of Associative Deletion

TemporalOverlay **restrict>expunge:** restrict-composite $\rightarrow$ expunge
  properties $\forall RE$ [E = restrict>expunge(R) $\supset$
    [instance(restrict-one,R.action) $\leftrightarrow$ instance(expunge-one,E)]
    $\wedge$ [instance(#restrict,R.action) $\leftrightarrow$ instance(#expunge,E)]]
  correspondences restrict-composite.old = expunge.old
    $\wedge$ restrict-composite.composite.op = expunge.key
    $\wedge$ restrict-composite.composite.two = expunge.input
    $\wedge$ restrict-composite.action.new = expunge.new
    $\wedge$ restrict-composite.action.in = expunge.in
    $\wedge$ restrict-composite.action.out = expunge.out

TemporalPlan **restrict-composite**
  roles .composite(function+two) .action(restrict)
  constraints complement(function+two>predicate(.composite),.action.criterion)

2.3 Keyed Discrimination

DataPlan **keyed-discrimination** *specialization* composed-functions
  properties $\forall Ds$ [instance(keyed-discrimination,D) $\supset$ instance(discrimination,composed>function(D,s))]
  roles .one(function) .two(function)
  constraints range-type(.two,finite-set)
    $\wedge$ $\forall Ts$ [($T \in$ range(.two)) $\supset$ set-type(set(T,s)) = domain-type(.one)]

2.4 Retrieval from a Keyed Discrimination

TemporalOverlay **discriminate+retrieve>retrieve:** keyed-discriminate+retrieve $\rightarrow$ retrieve
  correspondences
    discrimination>set(composed>function(keyed-discriminate+retrieve.composite))
      = retrieve.universe
    $\wedge$ keyed-discriminate+retrieve.if.key = retrieve.key
    $\wedge$ keyed-discriminate+retrieve.if.input = retrieve.input
    $\wedge$ keyed-discriminate+retrieve.if.output = retrieve.output
    $\wedge$ keyed-discriminate+retrieve.discriminate.in = retrieve.in
    $\wedge$ keyed-discriminate+retrieve.if.succeed = retrieve.succeed
    $\wedge$ keyed-discriminate+retrieve.if.fail = retrieve.fail

TemporalPlan **keyed-discriminate+retrieve**
roles .composite(keyed-discrimination) .discriminate(@function) .if(retrieve)
constraints .composite.one = .if.key ∧ .composite.two = .discriminate.op
∧ .discriminate.input = .if.input
∧ .discriminate.output = .if.universe
∧ cflow(.discriminate.out,.if.in)

2.5 Associative Deletion from a Keyed Discrimination

TemporalOverlay **discriminate+expunge+update>expunge:**
keyed-discriminate+expunge+update → expunge
properties ∀*DE* [*E* = discriminate+expunge+update>expunge(*D*) ⊃
[instance(expunge-one,*D*.action) ↔ instance(expunge-one,*E*)]
∧ [instance(#newvalue,*D*.update) ↔ instance(#expunge,*E*)]]
correspondences
discrimination>set(composed>function(keyed-discriminate+expunge+update.old))
= expunge.old
∧ discrimination>set
(composed>function(keyed-discriminate+expunge+update.new))
= expunge.new
∧ keyed-discriminate+expunge+update.action.input = expunge.input
∧ keyed-discriminate+expunge+update.action.key = expunge.key
∧ keyed-discriminate+expunge+update.discriminate.in = expunge.in
∧ keyed-discriminate+expunge+update.update.out = expunge.out

TemporalPlan **keyed-discriminate+expunge+update**
extension discriminate+action+update
roles .discriminate(@function) .action(expunge) .update(newvalue)
.old(keyed-discrimination) .new(keyed-discrimination)
constraints .discriminate.op = .old.two ∧ .action.key = .old.one
∧ .new.two = .update.new ∧ .new.one = .old.one

3. FUNCTIONS AND RELATIONS

Type **function** *specialization* object

Type **bijection** *specialization* function
definition instance(bijection,F) $\equiv$ [instance(function,F)
$\wedge$ $\forall xy$ [apply(F,x)=apply(F,y) $\supset$ $x=y$]]

Type **predicate** *specialization* function
definition instance(predicate,F) $\equiv$ [instance(function,F) $\wedge$ range-type(F)=boolean]

Function **domain-type:** function $\rightarrow$ type
properties $\forall Fx$ [apply(F,x)$\neq$undefined $\supset$ instance(domain-type(F),x)]

Function **range-type:** function $\rightarrow$ type
properties $\forall Fx$ [apply(F,x)$\neq$undefined $\supset$ instance(range-type(F),apply(F,x))]

Function **domain:** function $\rightarrow$ set
definition S=domain(F) $\equiv$ $\forall x$ [($x \in S$) $\leftrightarrow$ apply(F,x)$\neq$undefined]

Function **range:** function $\rightarrow$ set
definition S=range(F) $\equiv$ $\forall x$ [($x \in S$) $\leftrightarrow$ [$x$$\neq$undefined $\wedge$ $\exists y$ apply(F,y)=x]]

3.1 Input-Output and Test Specifications with Functions

IOspec **@function** / .op(function) .input(object) $\Rightarrow$ .output(object)
preconditions (.input $\in$ domain(.op))
postconditions apply(.op,.input)=.output

IOspec **newarg** / .old(function) .arg(object) .input(object) $\Rightarrow$ .new(function)
preconditions instance(domain-type(.old),.arg) $\wedge$ instance(range-type(.old).input)
postconditions apply(.new,.arg)=input
$\wedge$ $\forall xy$ [apply(.old,x)=y $\wedge$ $x$$\neq$.arg $\supset$ apply(.new,x)=y]
$\wedge$ domain-type(.new)=domain-type(.old) $\wedge$ range-type(.new)=range-type(.old)

IOspec **newvalue** / .old(function) .value(object) .input(object) $\Rightarrow$ .new(function)
preconditions instance(range-type(.old),.value) $\wedge$ instance(range-type(.old),.input)
postconditions $\forall x$ [apply(.old,x)=.value $\supset$ apply(.new,x)=.input]
$\wedge$ $\forall xy$ [apply(.old,x)=y $\wedge$ $y$$\neq$.value $\supset$ apply(.new,x)=y]
$\wedge$ $\forall xy$ [apply(.new,x)=y $\supset$ apply(.old,x)=y $\vee$ [apply(.old,x)=.value $\wedge$ y=.input]]
$\wedge$ domain-type(.new)=domain-type(.old) $\wedge$ range-type(.new)=range-type(.old)

Test **@predicate** / .criterion(predicate) .input(object)
condition apply(.criterion,.input)=true

3.2 Binary Functions

Type **binfunction** *specialization* object

Type **binrel** *specialization* binfunction
definition instance(binrel,F) $\equiv$ [instance(binfunction,F) $\wedge$ binrange-type(F)=boolean]

Function **argtype-one**: binfunction $\rightarrow$ type
properties $\forall Fxy$ [binapply(F,x,y)$\neq$undefined $\supset$ instance(argtype-one(F),x)]

Function **argtype-two**: binfunction $\rightarrow$ type
properties $\forall Fxy$ [binapply(F,x,y)$\neq$undefined $\supset$ instance(argtype-two(F),y)]

Function **binrange-type**: binfunction $\rightarrow$ type
properties $\forall Fxy$ [binapply(F,x,y)$\neq$undefined $\supset$ instance(binrange-type(F),binapply(F,x,y))]

IOspec **@binfunction** / .binop(binfunction) .one(object) .two(object) $\Rightarrow$ .output(object)
preconditions binapply(.binop,.one,.two.)$\neq$undefined
postconditions binapply(.binop,.one,.two)=.output

Test **@binrel** / .criterion(binrel) .one(object) .two(object)
condition binapply(.criterion,.one,.two)=true

3.3 Partial Orders

Type **partial-order** *specialization* binrel
definition instance(partial-order,F) $\equiv$ [instance(binrel,F)
$\wedge$ argtype-one(F)=argtype-two(F)
$\wedge$ $\forall x$ [$x\neq$undefined $\supset$ binapply(F,x,x)=true]
$\wedge$ $\forall xy$ [binapply(F,x,y)=true $\supset$ binapply(F,y,x)= false]
$\wedge$ $\forall xyz$ [binapply(F,x,y)=true $\wedge$ binapply(F,y,z)=true $\supset$ binapply(F,x,z)=true]]

Type **total-order** *specialization* partial-order
definition instance(total-order,R) $\equiv$ [instance(partial-order,R)
$\wedge$ $\forall xy$ [[instance(argtype-one,x) $\wedge$ instance(argtype-two,y)]
$\supset$ [binapply(R,x,y)=true $\vee$ binapply(R,y,x)=true]]]

Function **top**: partial-order $\rightarrow$ object
definition x=top(R) $\equiv$ $\forall y$ binapply(R,y,x)=true

Function **bottom**: partial-order $\rightarrow$ object
definition x=bottom(R) $\equiv$ $\forall y$ binapply(R,x,y)=true

3.4 Algebraic Binary Functions

Type **algebraic-binfunction** *subtype* binfunction
definition $\text{instance}(\text{algebraic-binfunction},F) \equiv [\,\text{instance}(\text{binfunction},F)$
$\wedge\ \text{argtype-one}(F)=\text{argtype-two}(F) \wedge \text{argtype-two}(F)=\text{binrange-type}(F)\,]$

Function **identity:** algebraic-binfunction → object
definition $e=\text{identity}(F) \equiv [\,\text{instance}(\text{binrange-type}(F),E)$
$\wedge\ \forall x\,[\,\text{instance}(\text{binrange-type}(F),x) \supset \text{binapply}(F,x,E)=x \wedge \text{binapply}(F,e,x)=x\,]\,]$

Predicate **commutative:** algebraic-binfunction → boolean
definition $\text{commutative}(F) \equiv \forall xy\ \text{binapply}(F,x,y)=\text{binapply}(F,y,x)$

Predicate **associative:** algebraic-binfunction → boolean
definition $\text{associative}(F) \equiv \forall xyz\ \text{binapply}(F,\text{binapply}(F,x,y),z)=\text{binapply}(F,x,\text{binapply}(F,y,z))$

3.5 Aggregative Binary Functions

Type **aggregative-binfunction** *subtype* algebraic-binfunction
definition $\text{instance}(\text{aggregative-binfunction},F) \equiv [\,\text{instance}(\text{algebraic-binfunction},F)$
$\wedge\ \text{associative}(F) \wedge \text{commutative}(F) \wedge \text{identity}(F)\neq\text{undefined}\,]$

Binfunction **plus:** integer × integer → integer
properties $\text{instance}(\text{aggregative-binfunction},\text{plus}) \wedge \text{identity}(\text{plus})=0$

Binfunction **times:** integer × integer → integer
properties $\text{instance}(\text{aggregative-binfunction},\text{times}) \wedge \text{identity}(\text{times})=1$

Binfunction **union:** set × set → set
properties $\text{instance}(\text{aggregative-binfunction},\text{union}) \wedge \text{empty}(\text{identity}(\text{union}))$
definition $U=\text{union}(S,T) \equiv \forall x\,[\,(x \in U) \leftrightarrow [\,(x \in S) \vee (x \in T)\,]\,]$

Binfunction **intersection:** set × set → set
properties $\text{instance}(\text{aggregative-binfunction},\text{intersection}) \wedge \text{universal}(\text{identity}(\text{intersection}))$
definition $U=\text{intersection}(S,T) \equiv \forall x\,[\,(x \in U) \leftrightarrow [\,(x \in S) \wedge (x \in T)\,]\,]$

Binfunction **greater:** integer × integer → integer
properties $\text{instance}(\text{aggregative-binfunction},\text{greater}) \wedge \text{identity}(\text{greater})=\text{minus-infinity}$
$\wedge\ \forall s\ \text{binrel}{>}\text{binchoice}(\text{le},s)=\text{greater}$
definition $k=\text{greater}(i,j) \equiv [\,[\,j=k \leftrightarrow \text{le}(i,j)\,] \wedge [\,i=k \leftrightarrow \text{le}(j,i)\,]\,]$

Binfunction **lesser:** integer × integer ⇢ integer
properties instance(aggregative-binfunction,lesser) ∧ identity(lesser) = infinity
∧ ∀*s* binrel>binchoice(ge,*s*) = lesser
definition *k* = lesser(*i*,*j*) ≡ [[*j* = *k* ↔ ge(*i*,*j*)] ∧ [*i* = *k* ↔ ge(*j*,*i*)]]

3.6 Composed Functions

DataPlan **composed-functions**
roles .one(function) .two(function)
constraints range-type(.one) = domain-type(.two) ∧ subset(range(.one),domain(.two))

DataPlan **hashing** *specialization* composed-functions
roles .one(function) .two(irredundant-sequence)

DataOverlay **composed>function:** composed-functions → function
definition *F* = composed>function(*C*,*s*) ≡
[domain-type(*F*) = domain-type(function(composed-functions(*C*,*s*).one,*s*))
∧ range-type(*F*) = range-type(function(composed-functions(*C*,*s*).two,*s*))
∧ ∀*x* apply(*F*,*x*) = apply(function(composed-functions(*C*,*s*).two,*s*),
apply(function(composed-functions(*C*,*s*).one,*s*)*x*))]

TemporalOverlay **composed>@function:** composed-applies → @function
correspondences composed-@functions.one.input = @function.input
∧ composed>function(composed-@functions.composite) = @function.op
∧ composed-@functions.two.output = @function.output
∧ composed-@functions.one.in = @function.in
∧ composed-@functions.two.out = @function.out

TemporalPlan **composed-@functions**
roles .composite(composed-functions) .one(@function) .two(@function)
constraints .composite.one = .one.op ∧ .composite.two = .two.op
∧ .one.output = .two.input ∧ cflow(.one.out,.two.in)

TemporalOverlay **newvalue-composite>newvalue:** newvalue-composite → newvalue
properties ∀*P* [instance(newvalue-composite(*P*)) ⊃
[instance(#newvalue,*P*.action) ↔ instance(#newvalue,newvalue-composite>newvalue(*P*)]]]
correspondences newvalue-composite.action.value = newvalue.value
∧ newvalue-composite.action.input = newvalue.input
∧ composed>function(newvalue-composite.old) = newvalue.old
∧ composed>function(newvalue-composite.new) = newvalue.new
∧ newvalue-composite.action.in = newvalue.in
∧ newvalue-composite.action.out = newvalue.out

TemporalPlan **newvalue-composite**
roles .action(newvalue) .old(composed-functions) .new(composed-functions)
constraints .old.one = .new.one ∧ .action.old = .old.two ∧ action.new = .new.two

3.7 Updating a Bijection

TemporalOverlay **newarg>newvalue:** newarg-bijection → @function+newvalue
properties ∀*N* [instance(newarg-bijection,*N*) ⊃
[instance(#newarg,*N*) ↔ instance(#newvalue,newarg>newvalue(*N*).update)]]
correspondences newarg-bijection.old = @function+newvalue.update.old
∧ newarg-bijection.arg = @function+newvalue.action.input
∧ newarg-bijection.input = @function+newvalue.update.input
∧ newarg-bijection.new = @function+newvalue.update.new
∧ newarg-bijection.in = @function+newvalue.action.in
∧ newarg-bijection.out = @function+newvalue.update.out

IOspec **newarg-bijection** / .old(bijection) .arg(object) .input(object)
⇒ .output(bijection)
specialization newarg

TemporalPlan **@function+newvalue**
roles .action(@function) .update(newvalue)
constraints instance(bijection,.action.op) ∧ .action.op = .update.old
∧ .action.output = .update.value ∧ cflow(.action.out,.update.in)

3.8 Binary Relations as Predicates[1]

DataOverlay **binrel+two>predicate:** binrel+two → predicate
definition *P* = binrel+two>predicate(*B*,*s*) ≡
∀*x* [apply(*P*,*x*) = true ↔ binapply(binrel(binrel+two(*B*,*s*).op,*s*),*x*,binrel+two(*B*,*s*).two) = true]

DataPlan **binrel+two**
roles .op(binrel) .two(object)
constraints instance(argtype-two(.op),.two)

1. See Fig. A-3.

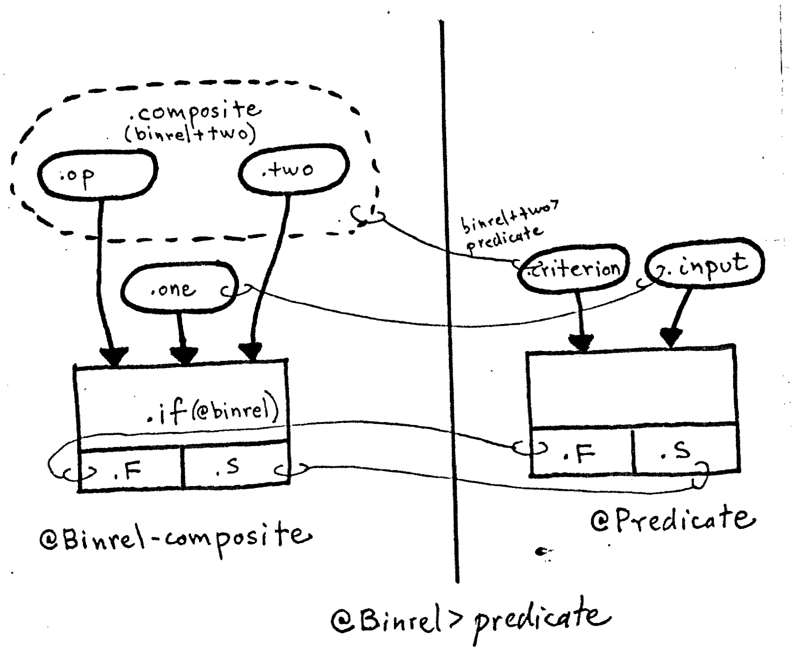


Figure A-3. Testing a Predicate Implemented as a Binary Relation.

TemporalOverlay **@binrel>predicate:** @binrel-composite → @predicate
correspondences
binrel+two>predicate(@binrel-composite.composite) = @predicate.criterion
∧ @binrel-composite.if.one = @predicate.input
∧ @binrel-composite.if.in = @predicate.in
∧ @binrel-composite.if.succeed = @predicate.succeed
∧ @binrel-composite.if.fail = @predicate.fail

TemporalPlan **@binrel-composite**
roles .composite(binrel+two) .if(@binrel)
constraints .composite.op = .if.criterion ∧ .composite.two = .if.two

DataOverlay **integer>predicate:** integer → predicate
definition P = integer>predicate(i,s) ≡ ∀j [apply(P,j) = true ↔ j = integer(i,s)]
properties ∀Bs [[binrel+two(B,s).op = eq ∧ instance(integer,binrel+two(B,s).two)]
⊃ binrel+two>predicate(B,s) = integer>predicate(B.two,s)]

3.9 Functions as Binary Relations

Type **many-to-one** *subtype* binrel
definition instance(many-to-one,R) ≡ [instance(binrel,R)
∧ ∀xyz [binapply(R,x,y) = true ∧ binapply(R,x,z) = true ⊃ y = z]]

DataOverlay **function>binrel:** function → many-to-one
definition R = function>binrel(F,s) ≡ ∀xy [apply(function(F,s),x) = y ↔ binapply(R,x,y) = true]

3.10 Functions as Predicates[1]

DataOverlay **function+two>predicate:** function+two → predicate
definition P = function+two>predicate(C,s) ≡
∀x [apply(P,x) = true ↔ apply(function(function+two(C,s).op,s),x) = function+two(C,s).two]

DataPlan **function+two**
roles .op(function) .two(object)
constraints instance(range-type(.op),.two)

1. See Fig. A-4.

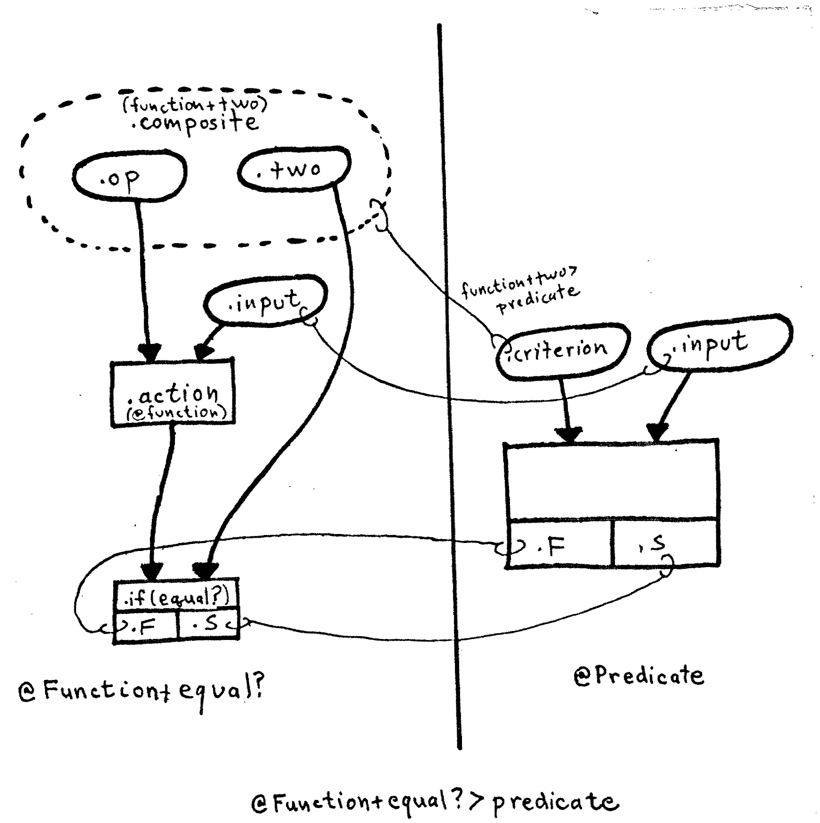


Figure A-4. Testing a Predicate Implemented as a Function.

TemporalOverlay **@function+equal?>predicate:** @function+equal → @predicate
correspondences function+two>predicate(@function+equal.composite) = @predicate.criterion
∧ @function+equal.action.input = @predicate.input
∧ @function+equal.action.in = @predicate.in
∧ @function+equal.if.succeed = @predicate.succeed
∧ @function+equal.if.fail = @predicate.fail

TemporalPlan **@function+equal?**
roles .composite(function+two) .action(@function) .if(equal?)
constraints .composite.op = .action.op ∧ .composite.two = .if.two
∧ .action.output = .if.one ∧ cflow(.action.out,.if.in)

Test **equal?** / .one(object) .two(object)
condition .one = .two

3.11 Function and Predicate Composites[1]

DataOverlay **function+predicate>predicate:** function+predicate → predicate
definition P = function+predicate>predicate(C,s) ≡
∀x [apply(P,x) = true ↔ apply(predicate(function+predicate(C,s).criterion,s)
,apply(function(function+predicate(C,s).op,s),x)) = true]

DataPlan **function+predicate**
roles .op(function) .criterion(predicate)
constraints range-type(.op) = domain-type(.criterion)

TemporalOverlay **@function+predicate>predicate:** @function+predicate → @predicate
correspondences
function+predicate>predicate(@function+predicate.composite) = @predicate.criterion
∧ @function+predicate.action.input = @predicate.input
∧ @function+predicate.action.in = @predicate.in
∧ @function+predicate.if.succeed = @predicate.succeed
∧ @function+predicate.if.fail = @predicate.fail

TemporalPlan **@function+predicate**
roles .composite(function+predicate) .action(@function) .if(@predicate)
constraints .composite.op = .action.op ∧ .composite.criterion = .if.criterion
∧ .action.output = .if.input ∧ cflow(.action.out,.if.in)

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1. See Fig. A-5

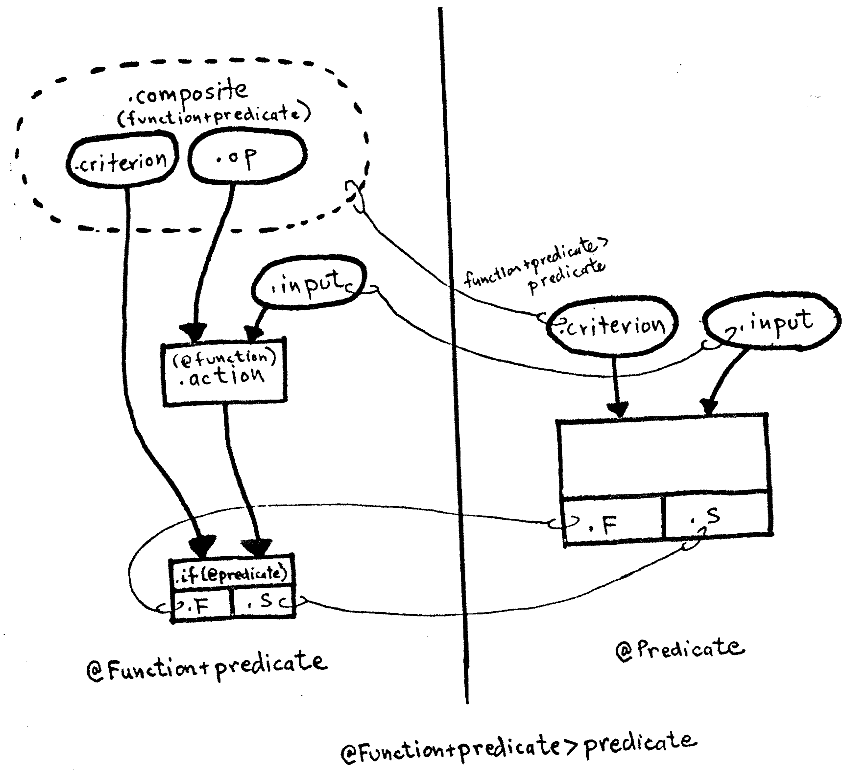

Figure A-5. Testing a Predicate Implemented as a Function and Predicate Composite.

3.12 Complementary Predicates

DataOverlay **complement:** predicate $\rightarrow$ predicate
properties $\forall xys$ [complement(x,s) = complement(y,s) $\supset$ predicate(x,s) = predicate(y,s)]
definition Q = complement(P,s) $\equiv$ $\forall x$ [apply(Q,x) = true $\leftrightarrow$ apply(predicate(P,s),x) = false]

TemporalOverlay **@predicate>complement:** @predicate $\rightarrow$ @predicate
properties $\forall xys$ [@predicate>complement(x,s) = @predicate>complement(y,s) $\supset$ $x = y$]
definition S = @predicate>complement(T) $\equiv$
[S.criterion = complement(T.criterion, T.in)
$\wedge$ S.input = T.input $\wedge$ S.in = T.in
$\wedge$ S.succeed = T.fail $\wedge$ S.fail = T.succeed]

3.13 Choice Functions[1]

Type **binchoice** *subtype* algebraic-binfunction
definition instance(binchoice,F) $\equiv$ [instance(binfunction,F)
$\wedge$ $\forall xy$ [binapply(F,x,y) = x $\vee$ binapply(F,x,y) = y]]

DataOverlay **binrel>binchoice:** binrel $\rightarrow$ binchoice
properties $\forall RFs$ [[F = binrel>binchoice(R,s) $\wedge$ instance(partial-order,binrel(R,s))]
$\supset$ [instance(aggregative-binfunction,F) $\wedge$ $\forall x$ [bottom(binrel(R,s),x) $\leftrightarrow$ x = identity(F))]]
definition F = binrel>binchoice(R,s) $\equiv$ $\forall xy$ [binapply(binrel(R,s),x,y) = true $\leftrightarrow$ binapply(F,x,y) = y]

TemporalOverlay **@binrel>choice:** @binrel+join $\rightarrow$ @choice
correspondences
binrel>binchoice(@binrel+join.if.criterion) = @choice.binop
$\wedge$ @binrel+join.end.output = @choice.output
$\wedge$ @binrel+join.if.in = @choice.in
$\wedge$ @binrel+join.end.out = @choice.out

IOspec **@choice** / .binop(binchoice) .one(object) .two(object) $\Rightarrow$ .output(object)
specialization @binfunction

TemporalPlan **@binrel+join** *specialization* cond
roles .if(@binrel) .then(in+out) .else(in+out) .end(join-output)
constraints .if.two = .end.succeed-input
$\wedge$ .if.one = .end.fail-input

1. See Fig. A-6.

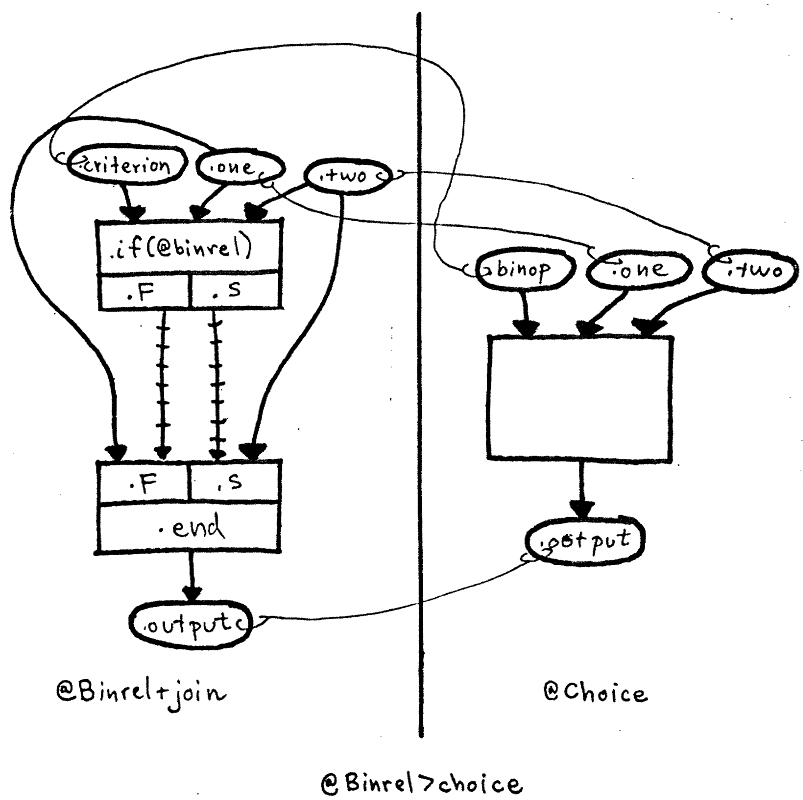

Figure A-6. Applying a Choice Function Implemented as a Binary Relation.

4. SEQUENCES

Type sequence *subtype* function
definition instance(sequence,F) $\equiv$ [domain-type(F,natural) $\wedge$ length(F)$\neq$undefined]

Type **irredundant-sequence** *subtype* bijection sequence

Type **finite-sequence** *subtype* sequence
definition instance(finite-sequence,S) $\equiv$ [instance(sequence,S) $\wedge$ finite(length(S))]

4.1 Relations on Sequences

Function **length:** sequence $\rightarrow$ cardinal
definition L=length(S) $\equiv$ $\forall i$ [apply(S,i)$\neq$undefined $\leftrightarrow$ [instance(natural,i) $\wedge$ le(i,L)]]

Binrel **index:** sequence $\times$ natural $\rightarrow$ boolean
definition index(S,i) $\equiv$ [instance(natural,i) $\wedge$ le(i,length(S))]

Function **first:** sequence $\rightarrow$ object
definition first(S) $\equiv$ apply(S,1)

Function **last:** finite-sequence $\rightarrow$ object
definition last(S) $\equiv$ apply(S,length(S))

Function **butlast:** finite-sequence $\rightarrow$ finite-sequence
definition T=butlast(S) $\equiv$ [length(S)=oneplus(length(T))
$\wedge$ $\forall ix$ [index(T,i) $\supset$ [apply(S,i)=x $\leftrightarrow$ apply(T,i)=x]]]

Function **reverse:** finite-sequence $\rightarrow$ finite-sequence
properties $\forall S$ reverse(reverse(S))=S
definition T=reverse(S) $\equiv$ [length(S)=length(T)
$\wedge$ $\forall i$ apply(S,i)=apply(T,oneplus(minus(length(S),i)))]

4.2 Input-Output Specifications with Sequences

IOspec **term** / .op(sequence) .input(natural) $\Rightarrow$ .output(object) *specialization* @function
preconditions index(.op,.input)

IOspec **newterm** / .old(sequence) .arg(natural) .input(object) $\Rightarrow$ .new(sequence)
specialization newarg
preconditions index(.old,.arg)

IOspec **#newterm** / .old(sequence) .arg(natural) .input(object) $\Rightarrow$ .new(sequence)
specialization newterm #newarg

IOspec **truncate** / .input(sequence) .criterion(predicate) $\Rightarrow$ .output(finite-sequence)
preconditions $\exists i$ apply(.criterion,apply(.input,i)) = true
postconditions
$\forall i$ [index(.output,i) $\leftrightarrow$ $\forall j$ [le(j,i) $\supset$ apply(.criterion,apply(.input,j)) = false]]
$\wedge$ $\forall i$ [index(.output,i) $\supset$ apply(.output,i) = apply(.input,i)]
$\wedge$ apply(.criterion,apply(.input,oneplus(length(.output)))) = true

IOspec **truncate-inclusive** / .input(seqence) .criterion(predicate)
$\Rightarrow$ .output(finite-sequence)
preconditions $\exists i$ apply(.criterion,apply(.input,i)) = true
postconditions
$\forall i$ [index(.output,i) $\leftrightarrow$ $\forall j$ [lt(j,i)
$\supset$ apply(.criterion,apply(.input,j)) = false]]
$\wedge$ $\forall i$ [index(.output,i) $\supset$ apply(.output,i) = apply(.input,i)]
$\wedge$ apply(.criterion,last(.output)) = true

IOspec **earliest** / .input(sequence) .criterion(predicate) $\Rightarrow$ .output(object)
preconditions $\exists i$ apply(.criterion,apply(.input,i)) = true
postconditions apply(.criterion,.output) = true
$\wedge$ $\exists i$ [apply(.input,i) = .output
$\wedge$ $\forall j$ [lt(j,i) $\supset$ apply(.criterion,apply(.input,j)) = false]]

IOspec **iterate** / .input(iterator) $\Rightarrow$ .output(sequence)
postconditions range-type(.output) = argtype-one(.input.op)
$\wedge$ first(.output) = .input.seed
$\wedge$ $\forall i$ [apply(.output,oneplus(i)) = successorn(i,generator>digraph(.input),.input.seed)]

IOspec **map** / .input(sequence) .op(function) $\Rightarrow$ .output(sequence)
preconditions subset(range(.input),domain(.op))
postconditions range-type(.output) = range-type(.op)
$\wedge$ length(.input) = length(.output)
$\wedge$ $\forall i$ [index(.input,i) $\supset$ apply(.output,i) = apply(.op,apply(.input,i))]

4.3 Segments

DataOverlay **segment>sequence**: segment $\rightarrow$ sequence
definition Q = segment>sequence(G,s) $\equiv$
[length(Q) = difference(natural(segment(G,s).upper,s),natural(segment(G,s).lower,s))
$\wedge$ $\forall i$ [index(Q,i) $\supset$ apply(Q,i) = apply(segment(G,s).base,plus(i,natural(segment(G,s).lower,s)))]]

DataPlan **segment**
roles .base(sequence) .lower(natural) .upper(natural)
constraints index(.base,.lower)
$\wedge$ index(.base,.upper) $\wedge$ le(.lower,.upper)

DataPlan **upper-segment** *specialization* segment
roles .base(sequence) .lower(natural) .upper(natural)
constraints .upper = length(.base)

DataPlan **lower-segment** *specialization* segment
roles .base(sequence) .lower(natural) .upper(natural)
constraints .lower = 1

5. LISTS

Type **list+nil** *uniontype* list nil

Type **finite-list** *specialization* list finite-single-recursion

Type **finite-list+nil** *uniontype* finite-list nil

DataPlan **irredundant-list** *specialization* list
definition instance(irredundant-list,L) $\equiv$ [instance(list,L)
$\wedge$ $\forall Ms$ [tail*(L,M) $\supset$ head(list(M,s)) $\neq$ L.head]
$\wedge$ instance(irredundant-list,list(L.tail,s))]

IOspec **push** / .old(list+nil) .input(object) $\Rightarrow$ .new(list)
specialization old+input+new
postconditions head(.new) = .input $\wedge$ tail(.new) = .old
$\wedge$ oneplus(length(.old)) = length(.new)

IOspec **pop** / .old(list) $\Rightarrow$ .new(list+nil) .output(object)
postconditions head(.old) = .output $\wedge$ tail(.old) = .new

IOspec **@head** / .op(function) .input(list) $\Rightarrow$ .output(object)
specialization @function
preconditions .op = head

IOspec **@tail** / .op(function) .input(list) $\Rightarrow$ .output(list+nil)
specialization @function
preconditions .op = tail

5.1 Upper Segment as List

DataOverlay **upper-segment>list:** upper-segment → list+nil
definition L=upper-segment>list(G,s) ≡
[[L=nil ↔ length(segment(G,s).base) = natural(segment(G,s).lower,s)]
∧ [instance(list,L) ⊃
∧ L.head = apply(sequence(segment(G,s).base,s),natural(segment(G,s).lower,s))
∧ ∃H [instance(upper-segment,H)
∧ sequence(segment(G,s).base,s) = sequence(segment(H,s).base,s)
∧ oneplus(natural(segment(G,s).lower,s)) = natural(segment(H,s).lower,s)
∧ L.tail = upper-segment>list(H,s)]]]

TemporalOverlay **bump+update>push:** bump+update → push
correspondences upper-segment>list(bump+update.old) = push.old
∧ bump+update.update.input = push.input
∧ upper-segment>list(bump+update.new) = push.new
∧ bump+update.bump.in = push.in
∧ bump+update.update.out = push.out

TemporalPlan **bump+update**
roles .bump(@oneminus) .update(newterm) .old(upper-segment) .new(upper-segment)
constraints cflow(.bump.out,.update.in)
∧ .old.lower = .bump.input
∧ .bump.output = .update.arg
∧ update.old = .old.base ∧ .update.new = .new.base
∧ .new.lower = .bump.output

IOspec **@oneminus** / .op(function) .input(integer) ⇒ .output(integer)
preconditions .op = oneminus

TemporalOverlay **fetch+bump>pop:** fetch+bump → pop
correspondences upper-segment>list(fetch+bump.old) = pop.old
∧ upper-segment>list(fetch+bump.new) = pop.new
∧ fetch+bump.fetch.output = pop.output
∧ fetch+bump.fetch.in = pop.in
∧ fetch+bump.bump.out = pop.out

TemporalPlan **fetch+bump**
roles .fetch(term) .bump(@oneplus) .old(upper-segment) .new(upper-segment)
constraints .old.base = .fetch.op ∧ .old.lower = .fetch.input
∧ .old.lower = .bump.input
∧ .new.base = .old.base ∧ .new.lower = .bump.output

IOspec **@oneplus** / .op(function) .input(integer) ⇒ .output(integer)
preconditions .op = oneplus

6. DIRECTED GRAPHS

DataPlan **digraph**
roles .nodes(set) .edge(binrel)

DataPlan **tree** *specialization* digraph
properties $\forall G$ instance(tree,G) $\supset \forall xy$ [root(G,x) $\wedge$ root(G,y) $\supset x=y$]
roles .nodes(set) .edge(binrel)
definition instance(tree,G) $\equiv$ [instance(digraph,G) $\wedge \exists x$ [root(G,x)] $\wedge \forall x$ [$\neg$successor*(G,x,x)]]

DataPlan **bintree** *specialization* tree
roles .nodes(set) .edge(binrel)
definition instance(bintree,T) $\equiv$ [instance(tree,T)
$\wedge \forall x$ [node(T,x) $\wedge \neg$terminal(T,x) $\supset$ size(successors(T,x))$=2$]]

DataPlan **thread** *specialization* tree
properties $\forall T$ instance(thread,T) $\supset$ [$\forall xy$ [terminal(T,x) $\wedge$ terminal(T,y) $\supset x=y$]
$\wedge \forall xyz$ [successor(T,x,y) $\wedge$ successor(T,z,y) $\supset x=z$]]
roles .nodes(set) .edge(many-to-one)

6.1 Relations on Directed Graphs

Binrel **node**: digraph $\times$ object $\rightarrow$ boolean
definition node(G,x) $\equiv$ ($x \in G$.nodes)

Trirel **successor**: digraph $\times$ object $\times$ object $\rightarrow$ boolean
definition successor(G,x,y) $\equiv$ [node(G,x) $\wedge$ node(G,y) $\wedge$ binapply(G.edge,x,y)$=$true]

Binfunction **successors**: digraph $\times$ object $\rightarrow$ set
definition $S=$successors(G,x) $\equiv \forall y$ [($y \in S$) $\leftrightarrow$ successor(G,x,y)]

Trirel **successor***: digraph $\times$ object $\times$ object $\rightarrow$ boolean
definition successor*(G,x,y) $\equiv \exists i$ successorn(i,G,x,y)

Quadrel **successorn**: natural $\times$ digraph $\times$ object $\times$ object $\rightarrow$ boolean
definition successorn(i,G,x,y) $\equiv$
[[$i=1$ $\wedge$ successor(G,x,y)]
$\vee \exists z$ [successor(G,x,z) $\wedge$ successorn(oneminus(i),G,z,y)]]

Binrel **root**: digraph $\times$ object $\rightarrow$ boolean
definition root(G,x) $\equiv \forall y$ [(node(G,y) $\wedge$ $x \neq y$) $\supset$ successor*(G,x,y)]

Binrel **terminal**: digraph $\times$ object $\rightarrow$ boolean
definition terminal(G,x) $\equiv$ [node(G,x) $\wedge \neg\exists y$ successor(G,x,y)]

Binrel **subgraph**: digraph $\times$ digraph $\rightarrow$ boolean
properties instance(partial-order,subgraph)
definition subgraph(G,H) $\equiv$

$$[\, \forall xy [\text{ successor}(G,x,y) \supset \text{successor}(H,x,y)]$$
$$\wedge\ \forall xy [\text{ node}(G,x) \wedge \text{node}(G,y) \wedge \text{successor}(H,x,y)$$
$$\supset \text{successor}(G,x,y)]\,]$$

6.2 Input-Output Specifications with Directed Graphs

IOspec **digraph-add** / .old(digraph) .input(object) $\Rightarrow$ .new(digraph)
specialization old+input+new
postconditions (.input $\in$ .new.nodes)

$$\wedge\ \forall xy [\, [\, x \neq .\text{input} \wedge y \neq .\text{input}$$
$$\wedge\ \neg\text{successor}(.\text{new},x,.\text{input}) \wedge \neg\text{successor}(.\text{new},y,.\text{input})$$
$$\wedge\ \neg\text{successor}(.\text{new},.\text{input},x) \wedge \neg\text{successor}(.\text{new},.\text{input},y)]$$
$$\supset [\text{ successor}(.\text{new},x,y) \leftrightarrow \text{successor}(.\text{old},x,y)]\,]$$

IOspec **digraph-remove** / .old(digraph) .input(object) $\Rightarrow$ .new(digraph)
specialization old+input+new
postconditions (.input $\notin$ .new.nodes)

$$\wedge\ \forall xy [\, x \neq .\text{input} \wedge y \neq .\text{input} \supset [\text{ successor}(.\text{new},x,y) \leftrightarrow \text{successor}(.\text{old},x,y)]\,]$$
$$\wedge\ \forall x [\text{ successor}(.\text{old},x,.\text{input})$$
$$\supset \forall y [\text{ successor}(.\text{new},x,y) \leftrightarrow \text{successor}(.\text{old},.\text{input},y)]\,]$$

IOspec **digraph-find** / .universe(digraph) .criterion(predicate) $\Rightarrow$ .output(object)
preconditions $\exists x$ [node(.universe,x) $\wedge$ apply(.criterion,x) = true]
postconditions node(.universe,.output) $\wedge$ apply(.criterion,.output) = true

6.3 Generators

DataPlan **generator**
roles .seed(object) .op(binrel)
constraints instance(argtype-one(.op),.seed) $\wedge$ argtype-one(.op) = argtype-two(.op)

DataPlan **iterator** *specialization* generator
roles .seed(object) .op(many-to-one)

DataOverlay **generator>digraph**: generator $\rightarrow$ digraph
properties $\forall Rs$ root(generator>digraph(R,s),generator(R,s).seed)
correspondences generator.op = digraph.edge

$$\wedge\ \text{transitive-closure(generator)} = \text{digraph.nodes}$$

DataOverlay **transitive-closure:** generator → set
definition T= transitive-closure(R,s) ≡
$\forall x\,[(x \in T) \leftrightarrow$
$[x = \text{generator}(R,s).\text{seed} \vee \exists y\,[(y \in T) \wedge \text{apply}(\text{function}(\text{generator}(R,s).\text{op},s),y) = x]]]$

Datastructure **natural-iterator** *instance* iterator
components .seed = 1 .op = oneplus

Datastructure **natural-thread** *instance* thread
properties $\forall s$ generator>digraph(natural-iterator,s) = natural-thread
components .nodes = naturals .edge = function>binrel(oneplus)

DataOverlay **binary>generator:** binary-generator → generator
properties $\forall BGs$ [G= binary>generator(B,s) ⊃ instance(bintree,generator>digraph(G,s))]
correspondences binary-generator.seed = generator.seed
∧ binrel-union(function>binrel(binary-generator.left),
function>binrel(binary-generator.right)) = generator.op

Binfunction **binrel-union:** binrel × binrel → binrel
definition T= binrel-union(R,S) ≡
$\forall xy\,[\text{binapply}(T,x,y) = \text{true} \leftrightarrow [\text{binapply}(R,x,y) = \text{true} \vee \text{binapply}(S,x,y) = \text{true}]]$

6.4 Truncated Directed Graphs

DataPlan **truncated-digraph**
roles .base(digraph) .criterion(predicate)
constraints $\forall x$ [node(.base,x) ⊃ [apply(.criterion,x) = true
∨ $\exists y$ [successor*(.base,x,y) ∧ apply(.criterion,y) = true]
∨ $\exists y$ [successor*(.base,y,x) ∧ apply(.criterion,y) = true]]]

DataPlan **truncated-tree** *specialization* truncated-digraph
roles .base(tree) .criterion(predicate)

DataPlan **truncated-thread** *specialization* truncated-tree
properties $\forall Ts$ [instance(truncated-thread,T) ⊃
$\exists x$ [node(digraph(T.base,s),x) ∧ apply(predicate(T.criterion,s),x) = true]]
roles .base(thread) .criterion(predicate)

6.5 Finite Subgraphs

DataOverlay **truncated>digraph:** truncated-digraph → finite-digraph
properties $\forall$ *TFs* [*F* = truncated>digraph-inclusive(*T*,*s*) ⊃
[instance(thread,digraph(truncated-digraph(*T*,*s*).base,*s*)) ↔ instance(thread,*F*)]
∧ [instance(tree,digraph(truncated-digraph(*T*,*s*).base,*s*)) ↔ instance(tree,*F*)]]
definition *F* = truncated>digraph(*T*,*s*) ≡
[subgraph(*F*,digraph(truncated-digraph(*T*,*s*).base,*s*))
∧ ∀*x* [node(*F*,*x*) ↔ [node(digraph(truncated-digraph(*T*,*s*).base,*s*),*x*)
∧ ∃*y* [successor*(digraph(truncated-digraph(*T*,*s*).base,*s*),*x*,*y*)
∧ apply(predicate(truncated-digraph(*T*,*s*).criterion,*s*),*y*) = true]
∧ ¬∃*z* [successor*(digraph(truncated-digraph(*T*,*s*).base,*s*),*z*,*x*)
∧ apply(predicate(truncated-digraph(*T*,*s*).criterion,*s*),*z*) = true]]]]

DataOverlay **truncated>digraph-inclusive:** truncated-digraph → finite-digraph
properties ∀ *TFs* [*F* = truncated>digraph-inclusive(*T*,*s*) ⊃
[instance(thread,digraph(truncated-digraph(*T*,*s*).base,*s*)) ↔ instance(thread,*F*)]
∧ [instance(tree,digraph(truncated-digraph(*T*,*s*).base,*s*)) ↔ instance(tree,*F*)]]
definition *F* = truncated>digraph-inclusive(*T*,*s*) ≡
[subgraph(*F*,digraph(truncated-digraph(*T*,*s*).base,*s*))
∧ ∀*x* [node(*F*,*x*) ↔ [node(truncated>digraph(*T*,*s*),*x*)
∨ ∃*y* [node(truncated>digraph(*T*,*s*),*y*)
∧ successor(digraph(truncated-digraph(*T*,*s*).base,*s*),*y*,*x*)
∧ apply(predicate(truncated-digraph(*T*,*s*).criterion,*s*),*x*) = true]]]]

DataPlan **finite-digraph** *specialization* digraph
roles .nodes(finite-set) .edge(binrel)

6.6 Trailing Plans

TemporalPlan **trailing** *extension* single-recursion
roles .current(object) .previous(object) .tail(trailing)
constraints .current = .tail.previous

TemporalPlan **trailing-search** *extension* trailing iterative-search
roles .current(object) .previous(object) .exit(cond) .tail(trailing-search)
constraints instance(join-two-outputs,.exit.end)
∧ .current = .exit.if.input ∧ .previous = .exit.end.succeed-input-two
∧ .tail.exit.end.output-two = .exit.end.fail-input-two

6.7 Trailing Generation and Search

TemporalPlan **trailing-generation+search** *extension* iterative-generation trailing-search
roles .current(object) .previous(object) .exit(cond) .action(@function)
.tail(trailing-generation+search)
constraints .current = .action.output ∧ .previous = .action.input

IOspec **internal-thread-find** / .universe(thread) .criterion(predicate)
⇒ .output(object) .previous(object)
extension digraph-find
preconditions ∀x [root(.universe,x) ⊃ apply(.criterion,x) = false]
postconditions successor(.universe,.previous,.output)

TemporalOverlay **trailing-generation+search>find :** trailing-iteration+search → internal-thread-find
correspondences
generator>digraph(temporal-iterator(trailing-generation+search))
= internal-thread-find.universe
trailing-generation+search.exit.if.criterion = internal-thread-find.criterion
trailing-generation+search.exit.end.output = internal-thread-find.output
trailing-generation+search.exit.end.two = internal-thread-find.previous
trailing-generation+search.action.in = internal-thread-find.in
trailing-generation+search.exit.out = internal-thread-find.out

6.8 Splicing Out of a Thread

TemporalPlan **spliceout**
roles .old(iterator) .new(iterator) .bump(@function) .splice(newarg)
constraints .old.op = .bump.op ∧ .new.op = .splice.out ∧ .old.seed = .new.seed
∧ .bump.output = .splice.input
∧ successor(generator>digraph(.old),.splice.arg,.bump.input)

IOspec **internal-thread-remove** / .old(thread) .input(object) ⇒ .new(thread)
specialization digraph-remove old+input+new
preconditions ¬root(.old,.input)

TemporalOverlay **spliceout>remove :** spliceout → internal-thread-remove
properties ∀S [instance(#spliceout,S) ↔ instance(#internal-thread-remove,spliceout>remove(S))]
correspondences
generator>digraph(spliceout.old) = internal-thread-remove.old
∧ spliceout.bump.input = internal-thread-remove.input
∧ generator>digraph(spliceout.new) = internal-thread-remove.new
∧ spliceout.bump.in = internal-thread-remove.in
∧ spliceout.splice.out = internal-thread.remove.out

TemporalPlan #**spliceout** *specialization* spliceout
roles .old(iterator) .new(iterator) .bump(@function) .splice(#newarg)

IOspec #**internal-thread-remove** / .old(thread) .input(object) ⇒ .new(thread)
specialization internal-thread-remove #old+input+new

6.9 Splicing Into a Thread[1]

TemporalPlan **splicein**
roles .old(iterator) .new(iterator) .one(newarg) .two(newarg)
constraints .one.arg = .two.input ∧ one.new = .two.old
∧ successor(generator>digraph(.old),.two.arg,.one.input)

IOspec **internal-thread-add** / .old(thread) .input(object) ⇒ .new(thread)
specialization digraph-add old+input+new
postconditions ¬root(.new,.input)
∧ ∀*x* [successor(.new,*x*,.input)
⊃ ∀*y* [successor(.old,*x*,*y*) ↔ successor(.new,.input,*y*)]]

TemporalOverlay **splicein>add:** splicein → internal-thread-add
properties ∀*S* [instance(#splicein,*S*) ↔ instance(#internal-thread-add,splicein>add(*S*))]
correspondences
generator>digraph(splicein.old) = internal-thread-add.old
∧ splicein.one.arg = internal-thread-add.input
∧ generator>digraph(splicein.new) = internal-thread-add.new
∧ find+splicein.one.in = internal-thread-add.in
∧ find+splicein.two.two.out = internal-thread-add.out

TemporalPlan #**splicein** *specialization* splicein
roles .old(iterator) .new(iterator) .one(#newarg) .two(#newarg)

IOspec #**internal-thread-add** / .old(thread) .input(object) ⇒ .new(thread)
specialization internal-thread-add #old+input+new

6.10 Labelled Directed Graphs

DataPlan **labelled-digraph**
roles .spine(digraph) .label(function)
constraints subset(.spine.nodes,domain(.label))

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1. See Fig. A-7.

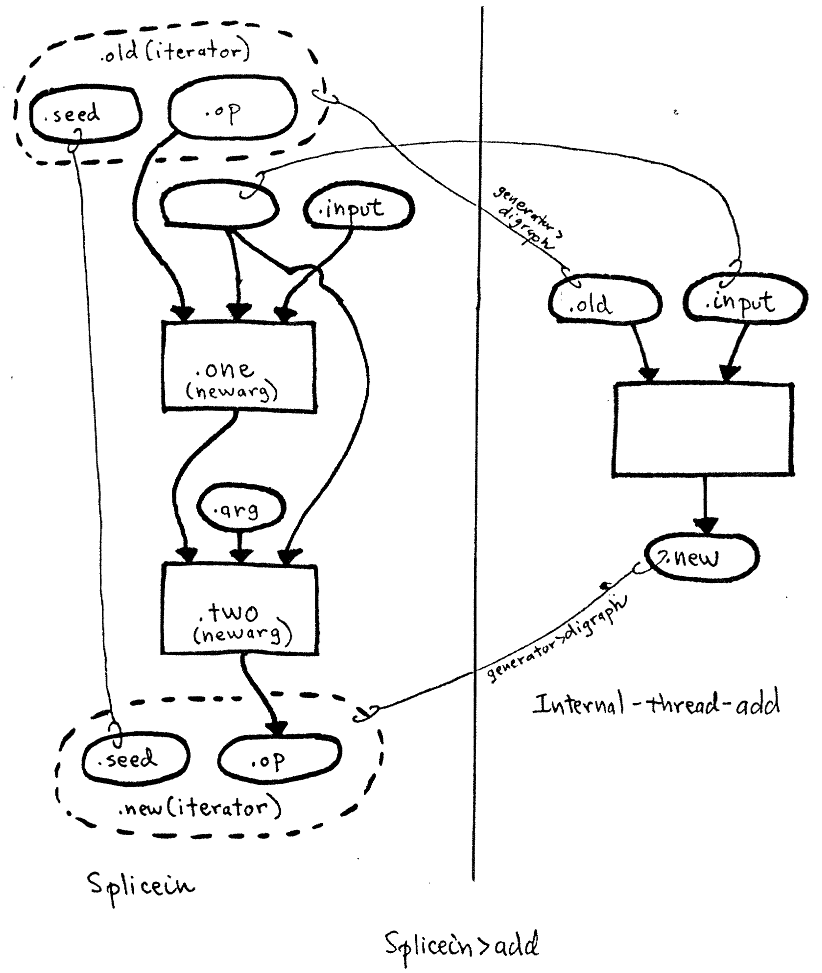

Figure A-7. Adding an Internal Node to a Thread by Splicing In.

DataPlan **labelled-thread** *specialization* labelled-digraph
roles .spine(thread) .label(function)

DataPlan **cdr-thread+car** *specialization* labelled-thread
properties $\forall Ps$ [instance(cdr-thread+car,P)
$\supset \exists x$ [root(thread(P.spine,s),x) $\wedge$ list>labelled-thread(dotted-pair>list(x,s),s) $= P$]]
roles .spine(cdr-thread) .label(function)
constraints .label = car

DataPlan **cdr-thread** *specialization* thread
roles .nodes(set) .edge(many-to-one)
constraints set-type(.nodes) = dotted-pair $\wedge$ $\forall s$ [binrel(.edge,s) = function>binrel(cdr,s)]

6.11 Trees as Partial Orders

DataOverlay **tree>order:** tree $\rightarrow$ partial-order+bottom
properties $\forall TRs$ [R = tree>order(T,s) $\supset$ [root(tree(T,s),bottom(R))
$\wedge$ [instance(thread,tree(T,s)) $\leftrightarrow$ instance(total-order,R)]]]
definition R = tree>order(T,s) $\equiv$ $\forall xy$ [binapply(R,x,y) = true $\leftrightarrow$ [$x = y$ $\vee$ successor*(tree(T,s),x,y)]]

Type **partial-order+bottom** *subtype* partial-order
definition instance(partial-order+bottom,R) $\equiv$ [instance(partial-order,R) $\wedge$ $\exists x$ bottom(R,x)]

6.12 Intervals

DataPlan **interval**
roles .base(total-order) .lower(object) .upper(object)
constraints binapply(.base,.lower,.upper) = true

DataOverlay **interval>truncated-thread:** interval $\rightarrow$ truncated-thread
properties $\forall ITs$ [T = interval>truncated-thread(I,s) $\supset$
interval(I,s).lower = bottom(tree>order(truncated>digraph(T,s),s))
$\wedge$ interval(I,s).upper = top(tree>order(truncated>digraph-inclusive(T,s),s))]
definition T = interval>truncated-thread(I,s) $\equiv$
[root(T,interval(I,s).lower)
$\wedge$ total-order(interval(I,s).base,s) = tree>order(thread(T.base,s),s)
$\wedge$ predicate(T.criterion,s) = integer>predicate(interval(I,s).upper,s)]

7. LINEAR STRUCTURES

DataOverlay **list>sequence:** list+nil → sequence
properties $\forall LQs$ [Q = list>sequence(L,s) ⊃
[length(list+nil(L,s)) = length(Q)
∧ [instance(irredundant-list,list+nil(L,s)) ↔ instance(irredundant-sequence,Q)]]]
∧ $\forall xys$ [list>sequence(x,s) = list>sequence(y,s) ⊃ list+nil(x,s) = list+nil(y,s)]
definition Q = list>sequence(L,s) ≡
[[length(Q) = 0 ↔ list+nil(L,s) = nil]
∧ [list(L,s) ≠ undefined ⊃
[first(Q) = list(L,s).head
∧ $\forall ix$ [apply(Q,i) = x ↔ $\exists M$ [tailn(oneminus(i),list(L,s)) = M ∧ list(M,s).head = x]]]]]

DataOverlay **sequence>labelled-thread:** finite-sequence → labelled-truncated-natural-thread
properties $\forall xys$ [sequence>labelled-thread(x,s) = sequence>labelled-thread(y,s)
⊃ sequence(x,s) = sequence(y,s)]
definition L = sequence>labelled-thread(Q,s) ≡
[function(L.label,s) = sequence(Q,s)
∧ $\exists T$ [digraph(L.spine,s) = truncated>digraph-inclusive(T,s)
∧ predicate(T.criterion,s) = integer>predicate(length(sequence(Q,s)),s)]]

DataOverlay **list>labelled-thread:** list → labelled-thread
definition T = list>labelled-thread(L,s) ≡
[$\forall x$ [[x = list(L,s) ∨ tail*(list(L,s),x)] ↔ ($x \in T$.spine.nodes)]
∧ T.spine.edge = tail ∧ T.label = head]

DataPlan **labelled-truncated-natural-thread** *specialization* labelled-thread
roles .spine(thread) .label(function)
constraints $\exists Ts$ [instance(truncated-thread,T) ∧ T.base = natural-thread
∧ .spine = truncated>digraph-inclusive(T,s)]

DataOverlay **sequence>thread:** irredundant-sequence → thread
properties $\forall QTs$ [T = sequence>thread(Q,s) ⊃
length(sequence(Q,s)) = size(set(T.nodes,s)) ∧ terminal(T,last(sequence(Q,s)))]
∧ $\forall xys$ [sequence>thread(x,s) = sequence>thread(y,s) ⊃ sequence(x,s) = sequence(y,s)]
definition T = sequence>thread(Q,s) ≡
[root(T,first(sequence(Q,s)))
∧ $\forall i$ [index(sequence(Q,s),i)
⊃ successor(T,apply(sequence(Q,s),i),apply(sequence(Q,s),oneplus(i)))]]

DataOverlay **list>thread:** irredundant-list → thread
properties $\forall xys$ [list>thread(x,s) = list>thread(y,s) ⊃ list(x,s) = list(y,s)]
definition T = list>thread(L,s) ≡ $\forall x$ [[list(L,s).head = x ↔ root(T,x)]
∧ $\forall ix$ [$\exists M$ [tailn(i,list(L,s),M) ∧ M.head = x] ↔ $\exists y$ [root(T,y) ∧ successorn(i,T,y,x)]]]]

8. FLAGS

DataPlan **flag**
roles .arg(object) .criterion(predicate)
constraints instance(domain-type(.criterion),.arg)

TemporalPlan **enflag+output**
roles .enflag(cond) .output(flag)
constraints instance(join-output,.enflag.end)
∧ .output.arg = .enflag.end.output
∧ apply(.output.criterion,.enflag.end.succeed-input) = true
∧ apply(.output.criterion,.enflag.end.fail-input) = false

TemporalPlan **enflag+deflag** *extension* enflag+output
roles .enflag(cond) .output(flag) .deflag(@predicate)
constraints .deflag.criterion = .enflag.output.criterion
∧ .enflag.end.output = .deflag.input
∧ cflow(enflag.end.out,.deflag.in)

TemporalOverlay **enflag+deflag>test**[1]**:** enflag+deflag → test
correspondences enflag+deflag.enflag.if.in = test.in
∧ enflag+deflag.deflag.succeed = test.succeed
∧ enflag+deflag.deflag.fail = test.fail

1. See Fig. A-8.

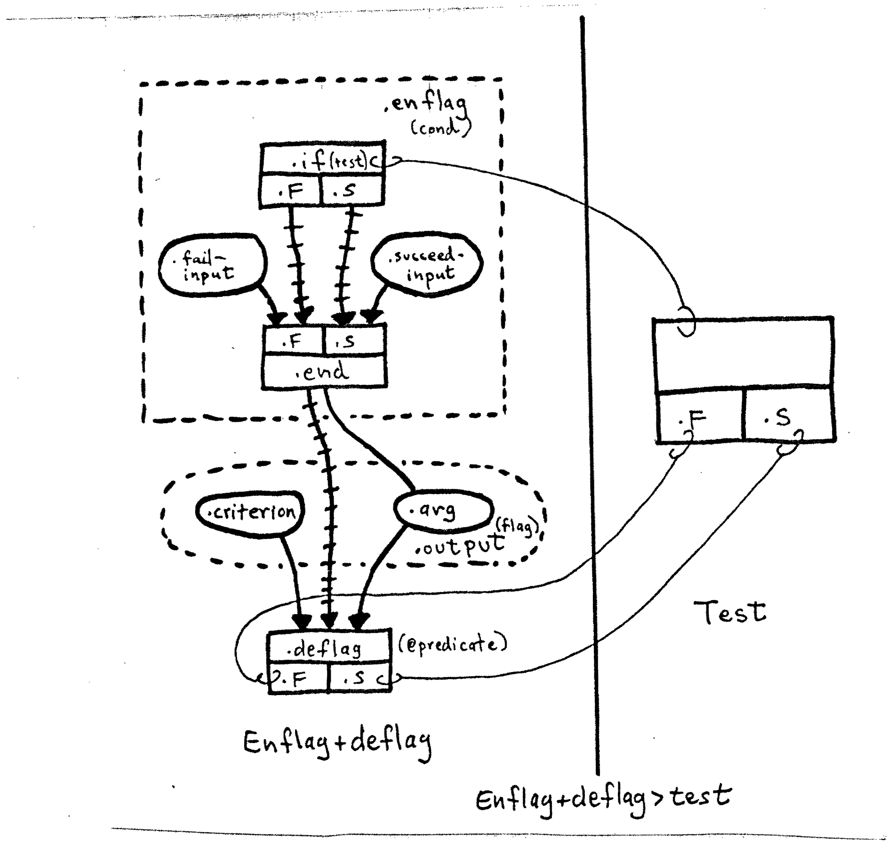

Figure A-8. Viewing Enflag and Deflag as a Test.

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