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FAN-BEAM RECONSTRUCTION METHODS

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ABSTRACT. In a previous paper a technique was developed for finding reconstruction algorithms for arbitrary ray-sampling schemes. The resulting algorithms use a general linear operator, the kernel of which depends on the details of the scanning geometry. Here this method is applied to the problem of reconstructing density distributions from arbitrary fan-beam data. The general fan-beam method is then specialized to a number of scanning geometries of practical importance. Included are two cases where the kernel of the general linear operator can be factored and rewritten as a function of the difference of coordinates only and the superposition integral consequently simplifies into a convolution integral. Algorithms for these special cases of the fan-beam problem have been developed previously by others. In the general case, however, Fourier transforms and convolutions do not apply, and linear space-variant operators must be used. As a demonstration, details of a fan-beam method for data obtained with uniform ray-sampling density are developed.

REVIEW.

In a previous paper [1], I developed a technique for finding reconstruction algorithms applicable to arbitrary ray-sampling schemes. This general method was applied to the problem of reconstruction from parallel-beam data with uneven spacing between rays and uneven spacing between projections. The results were based on Radon's famous integral [2],

$$f(r,\phi) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_{-\infty}^{\infty} \left(-\frac{1}{t}\right) \frac{\partial}{\partial \ell} p(\ell,\theta)\, d\ell\, d\theta \tag{1}$$

where $p(\ell,\theta)$ is the density integral or ray-sum measured along the ray inclined θ with respect to a vertical axis and passing within a distance ℓ from the center of the region being scanned (see figure 1). Further, $f(r,\phi)$ is the density at the point with polar coordinates (r,ϕ) in this region, while $t = \ell - r\cos(\theta - \phi)$ is the perpendicular distance between the ray and this point.

When ray sums in a given projection are spaced evenly in ℓ, and projections are spaced evenly in θ, a simple reconstruction method can be found directly from equation 1 by approximating both integrals by sums and approximating the partial derivative by an appropriate first difference.

UNIFORM SCANNING COORDINATES.

When spacing is uneven, it is helpful to introduce first new ray-sampling coordinates ξ and η, chosen so that successive rays in a generalized projection correspond to evenly spaced values of ξ, while successive projections correspond to evenly spaced values of η. Radon's integral can then be transformed to this new coordinate system using the Jacobian,

$$J = \frac{\partial \ell}{\partial \xi} \cdot \frac{\partial \theta}{\partial \eta} - \frac{\partial \ell}{\partial \eta} \cdot \frac{\partial \theta}{\partial \xi} \tag{2}$$

and equation 1 becomes,

$$f(r,\phi) = \frac{1}{4\pi^2} \iint (-\frac{1}{t}) \frac{\partial p}{\partial \ell} \; J(\xi,\eta) \; d\xi \; d\eta \tag{3}$$

It is possible to show that this can be rewritten as

$$f(r,\phi) = \frac{1}{4\pi^2} \iint (-\frac{1}{t}) \left[\frac{\partial p}{\partial \xi} \cdot \frac{\partial \theta}{\partial \eta} - \frac{\partial p}{\partial \eta} \cdot \frac{\partial \theta}{\partial \xi} \right] d\xi \; d\eta \tag{4}$$

It is not clear whether this forms a good basis for a reconstruction algorithm in the general case, since it seems to imply that computations must be carried out <u>across</u> projections as well as <u>within</u> projections.

GENERAL PARALLEL BEAM METHOD.

In the previous paper [1], the emphasis was on parallel-ray scanning, and, in this case, ℓ is a function of ξ only, while θ is a function of η only. The Jacobian then reduces to,

$$J = \frac{\partial \ell}{\partial \xi} \cdot \frac{\partial \theta}{\partial \eta} \tag{5}$$

and equation 4 simplifies as follows,

$$f(r,\phi) = \frac{1}{4\pi^2} \int [\int (- \frac{1}{t} \frac{\partial p}{\partial \xi} \, d\xi] \frac{\partial \theta}{\partial \eta} \, d\eta \tag{6}$$

Here $t = \ell - \ell'$, where $\ell = \ell(\xi)$, while $\ell' = \ell(\xi')$, and ξ' is the value of ξ associated with the ray that passes through the point (r,ϕ). The above can be conveniently split into an outer and an inner integral:

$$f(r,\phi) = \frac{1}{4\pi^2} \int g(\xi',\eta) \frac{\partial \theta}{\partial \eta} \, d\eta \tag{7}$$

$$g(\xi',\eta) = - \int \frac{1}{\ell(\xi) - \ell(\xi')} \frac{\partial}{\partial \xi} p(\xi,\eta) \, d\xi \tag{8}$$

If these integrals are approximated by sums, one obtains:

$$f(r,\phi) \simeq \frac{1}{4\pi^2} \sum_j g_j(\xi') \, \delta\theta_j \tag{9}$$

$$g_{i'j} = - \sum_{i} \frac{(p_{ij} - p_{(i-1)j})}{(\ell_i - \ell'_{i'})} \tag{10}$$

This straightforward set of equations is one result of the analysis in the previous paper (equations 29 and 40 in [1]). Here $\delta\theta_j = (\theta_{j+1} - \theta_{j-1})/2$ is the angular interval associated with the j^{th} projection, while $\ell'_i = (\ell_i + \ell_{i+1})/2$ is the value of ℓ corresponding to the center of the i^{th} beam. The left edge of the beam striking the i^{th} detector corresponds to ℓ_i and the right edge to ℓ_{i+1} (see figure 2). The density integral obtained from the i^{th} detector in the j^{th} projection is p_{ij}.

Finally, note that $g_j(\xi')$ has to be found by interpolation from the discrete set of values, $\{g_{i'j}\}$. If linear interpolation is to be used, one can work with the values $g_{i'j}$ and $g_{(i'+1)j}$, where

$$\ell'_i \leq \ell(\xi') < \ell'_{i+1} \tag{11}$$

RELATION TO CONVOLUTION-BACKPROJECTION METHOD.

By splitting the second sum and rearranging its terms, one arrives at an alternate form (equation 38 in [1]),

$$g_{i'j} = \frac{4}{(\ell_{i'+1} - \ell_{i'})} p_{i'j} - \sum_{i \neq i'} \frac{(\ell_{i+1} - \ell_i)}{(\ell_{i+1} - \ell_{i'})(\ell_i - \ell_{i'})} p_{ij} \quad (12)$$

That is, the sequence $\{g_{ij}\}$ is obtained from the sequence $\{p_{ij}\}$ by a general linear operator. This is similar to a convolution except that the weights or filter coefficients are spatially variant.

One has only to fix the width of the detectors, at $\delta\ell$ say, to be able to relate this result to the well-known convolutional-backprojection method. In this case,

$$\delta\ell \; g_{i'j} = [4p_{i'j} - \sum_{i \neq i'} \frac{4}{4(i - i')^2 - 1} p_{ij}] \quad (13)$$

This amounts to convolution of $\{p_{ij}\}$ with a filter function F_k, where

$$F_k = -\frac{4}{4k^2 - 1} \quad \text{for } k \neq 0 \text{ and } F_o = 4. \quad (14)$$

This happens to be the particular set of filter coefficients popularized by Shepp and Logan [3].

My previous paper contains other formulations for this problem as well as a simulation of the method for reconstruction from ray sums collected with uneven spacing [1]. The main point is that convolutions are inappropriate in the general case and must be replaced by spatially varying operators or superposition integrals.

RECONSTRUCTION FOR ARBITRARY RAY-SAMPLING SCHEMES.

In the general case, equation (4) does not seem to provide a good starting point for analysis. Instead it is helpful first to remove the partial derivative from Radon's integral by partial integration. This has to be done carefully since the inner integral is singular. It can be written as

$$\lim_{\varepsilon\to 0} \int_{-\infty}^{\ell'-\varepsilon} \left(-\frac{1}{t}\right) \frac{\partial}{\partial \ell} p(\ell,\theta)d\ell + \lim_{\varepsilon\to 0} \int_{\ell'+\varepsilon}^{+\infty} \left(-\frac{1}{t}\right)\frac{\partial}{\partial \ell} p(\ell,\theta)\, d\ell \qquad (15)$$

Integrating by parts one obtains (equation 8 in [1]),

$$f(r,\phi) = \frac{1}{4\pi^2} \int_0^{2\pi} \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} F_\varepsilon(t)p(\ell,\theta)d\ell\; d\theta \qquad (16)$$

where

$$F_\varepsilon(t) = 1/\varepsilon^2 \quad \text{for } |t| < \varepsilon \qquad (17a)$$

$$= -\,1/t^2 \text{ for } |t| \geq \varepsilon \qquad (17b)$$

Introduction of the transformation to uniform scanning coordinates (ξ,η) now leads to

$$f(r,\phi) = \frac{1}{4\pi^2} \int g(r,\phi,\eta)\, d\eta \tag{18}$$

$$g(r,\phi,\eta) = \lim_{\varepsilon \to 0} \int F\,(t) J(\xi,\eta) p(\xi,\eta)\, d\xi \tag{19}$$

This forms the basis for reconstruction methods for arbitrary ray-sampling schemes. In this paper the concern will be with techniques for reconstructing density distributions from ray sums collected using fan beams (see figure 3). Modern apparatus for computerized tomographic analysis typically produces projection data in this form and there is a practical need for accurate and rapid reconstruction methods for a variety of different schemes for sampling the fan beam. Such methods had been found previously for two special geometries [4, 5, 6]. Here techniques will be developed that can be used for arbitrary ray-sum collection schemes.

SOURCE POSITIONS DISTRIBUTED AROUND THE CIRCUMFERENCE OF A CIRCLE.

Let the source be located at $(D, \pi/2 + \beta)$, where D is the radius of the circle (see figure 1). Let a ray be emitted in a direction that makes an angle α with the source-to-origin line. Clearly α and β are as good for specifying a particular ray as l and θ are. For fan beams, these new parameters will be more directly useful, and so the relationships between the two sets of variables will be needed. From figure 1,

$$\ell = D \sin \alpha \qquad \text{and} \qquad \theta = \alpha + \beta \tag{20}$$

$$\alpha = \sin^{-1}(\ell/D) \qquad \text{and} \qquad \beta = \theta - \sin^{-1}(\ell/D) \tag{21}$$

If ξ and η are uniform scanning coordinates, then it is natural to let

$$\alpha = \alpha(\xi) \qquad \text{and} \qquad \beta = \beta(\eta) \tag{22}$$

where α and β are continuous, differentiable monotonic functions of ξ and η. Then (see equation 2),

$$J = \frac{\partial \ell}{\partial \xi} \frac{\partial \theta}{\partial \eta} \quad \text{since} \quad \frac{\partial \ell}{\partial \eta} = 0 \tag{23}$$

Further, since $\theta = \alpha + \beta$,

$$J = \frac{\partial \ell}{\partial \alpha} \frac{\partial \alpha}{\partial \xi} \frac{\partial \beta}{\partial \eta} = \frac{\partial \beta}{\partial \eta} \; D \cos \alpha \; \frac{\partial \alpha}{\partial \xi} \tag{24}$$

Now J is a factor in the inner integral (equation 19), but the first term of the above product can clearly be brought out of the inner integral and incorporated into the outer integral (equation 18). The last term of the product will depend on the way in which rays in a fan are sampled. This corresponds to the placement of detectors in the fan and depends on the scanning scheme used. We will study several cases after developing a few more tools that will be needed.

THE PERPENDICULAR DISTANCE FROM A POINT TO A RAY FOR FAN BEAMS.

From the diagram (figure 4) we can develop a useful new way of writing t, the distance between a given ray (α,β) and a given point (r,ϕ):

$$t = K \sin (\alpha - \alpha') \tag{25}$$

where, by the cosine rule for triangles (see figure 5),

$$K^2 = r^2 + D^2 + 2r D \sin (\beta - \phi) \tag{26}$$

Here α' is the value of α corresponding to a ray from the source which passes directly through the given point (r,ϕ). Note that K is simply the distance from the source to the point (r,ϕ) and thus clearly does not depend on α. From the diagram we can calculate α' as follows (see figure 5):

$$K \sin \alpha' = r \cos (\beta - \phi) \quad \text{and} \quad K \cos \alpha' = D + r \sin (\beta - \phi) \tag{27}$$

and so,

$$\tan \alpha' = \frac{r \cos (\beta - \phi)}{[D + r \sin (\beta - \phi)]} \tag{28}$$

SOME PROPERTIES OF THE FILTERING FUNCTION.

Note that if $c \neq 0$, then

$$F_\varepsilon(ct) = \frac{1}{c^2} F_{\varepsilon/c}(t) \tag{29}$$

This result can be easily checked by separately considering the cases $|ct| < \varepsilon$ and $|ct| \geq \varepsilon$. From this it follows that

$$\int_{-\infty}^{\infty} F_\varepsilon(ct)dt = 0 \tag{30}$$

and

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} F_\varepsilon(ct)\ a\ (t)\ dt = \frac{1}{c^2} \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} F_\varepsilon(t)\ a\ (t)\ dt \tag{31}$$

Furthermore if $|b(t)| \geq b_0 > 0$ is continuous and differentiable with respect to t,

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} F_\varepsilon[b(t)t]\ a(t)dt = \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} F_\varepsilon(t) \frac{a(t)}{b^2(t)}\ dt \tag{32}$$

These results are useful in deriving the general reconstruction formula for fan-beam scanning schemes.

RECONSTRUCTION ALGORITHM FOR ARBITRARY FAN BEAM GEOMETRIES.

Using the general result for reconstruction (equations 18 and 19) based on Radon's inversion formula [2] and using the expressions for t and J just derived, we have

$$f(r,\phi) = \frac{1}{4\pi^2}\int g'(r,\phi,\eta)\,\frac{\partial\beta}{\partial\eta}\,d\eta \tag{33}$$

$$g'(r,\phi,\eta) = \lim_{\varepsilon \to 0}\int F_\varepsilon[K \sin(\alpha - \alpha')]D \cos\alpha\,\frac{\partial\alpha}{\partial\xi}\,p(\xi,\eta)\,d\xi \tag{34}$$

According to a result just developed (equation 31), the factor K can be extracted from the argument of the filter function F. Since K does not depend on α it can be further removed from the inner integral (equation 34) and placed into the outer integral (equation 33). The inner integral then no longer contains terms which depend explicitly on r and ϕ, only terms which are a function of α'. So the above can be rewritten,

$$f(r,\phi) = \frac{1}{4\pi^2}\int g''(\alpha',\eta)\,(1/K^2)\,\frac{\partial\beta}{\partial\eta}\,d\eta \tag{35}$$

$$g''(\alpha',\eta) = \lim_{\varepsilon \to 0}\int F_\varepsilon[\sin(\alpha - \alpha')]\,D\cos\alpha\,\frac{\partial\alpha}{\partial\xi}\,p(\xi,\eta)\,d\xi \tag{36}$$

From the fact that the inner integral is a function of α', and not r and ϕ explicitly, we conclude that the reconstruction algorithm can be arranged efficiently. That is, for all fan beam schemes with source positions on the circumference of a circle and sampling of the fan independent of source position, one need not explicitly calculate the contribution of each ray to each point in the reconstruction.

We are now ready to develop specific reconstruction methods for a variety of fan-sampling schemes. Some special schemes will lead to simplifications which can be interpreted as pre-multiplication, convolution and post-multiplication. In general, however, the inner integral remains in the form of a general linear operator or superposition integral.

UNIFORM SAMPLING ALONG A LINE AT RIGHT ANGLES TO THE SOURCE-TO-ORIGIN LINE.

Rays are sampled evenly in λ (see figure 6), so it is natural to let

$$\xi = \lambda = D \tan \alpha \tag{37}$$

$$D \cos \alpha \left(\frac{\partial \alpha}{\partial \xi}\right) = \cos^3 \alpha \tag{38}$$

Also,

$$\sin \alpha = \frac{\xi}{\sqrt{D^2 + \xi^2}} \quad \text{and} \quad \cos \alpha = \frac{D}{\sqrt{D^2 + \xi^2}} \tag{39}$$

So

$$D \cos \alpha \left(\frac{\partial \alpha}{\partial \xi}\right) = \frac{D^3}{[D^2 + \xi^2]^{3/2}} \tag{40}$$

Further, if we let α' be the value of α corresponding to the ray through the point (r,ϕ) and ξ' the corresponding value of ξ, then

$$\sin \alpha' = \frac{\xi'}{\sqrt{D^2 + \xi'^2}} \quad \text{and} \quad \cos \alpha' = \frac{D}{\sqrt{D^2 + \xi'^2}} \tag{41}$$

So,

$$\sin (\alpha - \alpha') = \sin \alpha \cos \alpha' - \sin \alpha' \cos \alpha \tag{42}$$

$$\sin(\alpha - \alpha') = \frac{D}{\sqrt{D^2 + \xi^2}\sqrt{D^2 + \xi'^2}}(\xi - \xi') \tag{43}$$

Hence,

$$g''(\xi',\beta) = \lim_{\varepsilon \to 0} \int F_\varepsilon \left[\frac{1}{\sqrt{D^2 + \xi'^2}}(\xi - \xi')\frac{D}{\sqrt{D^2 + \xi^2}}\right] \frac{D^3}{[D^2 + \xi^2]^{3/2}} p(\xi,\beta)d\xi \tag{44}$$

We can move the multiplier of $(\xi - \xi')$ out of the filter function F to get:

$$g''(\xi',\beta) = (D^2 + \xi'^2) \lim_{\varepsilon \to 0} \int F_\varepsilon (\xi - \xi') \frac{D}{\sqrt{D^2 + \xi^2}} p(\xi,\beta)\, d\xi \tag{45}$$

Note that (from equation 37 or 41),

$$\xi' = D \tan \alpha' \tag{46}$$

and so

$$(D^2 + \xi'^2) = D^2 \sec^2 \alpha' \tag{47}$$

Also, combining this with the term $(1/K^2)$ in the outer integral one gets

$$\frac{(D^2 + \xi'^2)}{K^2} = \frac{D^2}{[K \cos \alpha']^2} = \frac{D^2}{[D + r \sin(\beta - \phi)]^2} \tag{48}$$

and so, finally,

$$f(r,\phi) = \frac{1}{4\pi^2} \int_0^{2\pi} g'''(\lambda', \beta) \frac{D^2}{[D + r \sin (\beta - \phi)]^2} d\beta \tag{49}$$

$$g'''(\lambda',\beta) = \lim_{\varepsilon \to 0} \int_{-\infty}^{+\infty} F_\varepsilon(\lambda - \lambda') \frac{D}{\sqrt{D^2 + \lambda^2}} p(\lambda,\beta) \, d\lambda \tag{50}$$

Here finally we have used the coordinate natural to this particular scanning scheme, namely λ, the distance from the origin at which the ray intersects the line drawn perpendicular to the source-to-origin line.

It is important to note that in this case the argument of the filter function F contains only the difference of the two parameters λ and λ'. The above is thus almost like a convolution, except that one has to pre-multiply the projection data $p(\lambda,\beta)$ by a factor depending on the position of the ray in the fan. Similarly, the convolved data, g''', is used in the outer integral after post-multiplication by a factor which depends on the position of the point (r,ϕ) in the fan currently being processed.

DISCRETE APPROXIMATION.

Finally, we have to approximate these integrals by sums because in practice only a finite set of ray sums is available:

$$f(r,\phi) \approx \frac{1}{4\pi^2} \sum_j g_j(\lambda') \frac{D^2}{[D + r \sin (\beta_j - \phi)]^2} \delta\beta_j \tag{51}$$

$$g_{i'j} = \sum_i F_{i-i'} \frac{D}{\sqrt{D^2 + \lambda_i^2}} P_{ij} \delta\lambda \tag{52}$$

Here

$$\lambda' = D \tan \alpha' = \frac{D r \cos (\beta_j - \phi)}{D + r \sin (\beta_j - \phi)} \tag{53}$$

In the above set of reconstruction equations, P_{ij} is the i^{th} ray sum in the j^{th} fan, while $\delta\lambda$ is the (fixed) interval between intersection of successive rays with the line perpendicular to the source-to-origin line. The angular interval associated with a particular fan is $\delta\beta_j$, where

$$\delta\beta_j = (\beta_{j+1} - \beta_{j-1})/2 \tag{54}$$

The filter factors are

$$F_k = -\frac{w_k}{(k\ \delta\lambda)^2} \qquad k \neq 0 \tag{55}$$

$$F_o = -2 \sum_{k=1}^{\infty} F_k \tag{56}$$

As mentioned in the paper on which this analysis is based, the weights w_k are chosen to provide good numerical approximations for the singular integral. Typical choices are:

1. $w_k = 2$ for k even, $w_k = 0$ for k odd

2. $w_k = 4k^2/(4k^2 - 1)$

3. $w_k = 1$

Note that the above operation can be viewed as a pre-multiplication of the ray sums by $D/[D^2 + \lambda_i^2]^{1/2}$, followed by a convolution, with a final post-multiplication by $D^2/[D + r \sin (\beta_j - \phi)]^2$. While the method is not strictly convolutional, it can be conveniently viewed in this way. The above is one of two special cases of the fan-beam problem that had previously been solved [4, 5]. An attempt was made here to use similar notation to simplify comparison.

We ought to specify how $g_j(\lambda')$ is found from $g_{i'j}$. As in the previous paper [1], we approximate $g_j(\lambda')$ by interpolation. If we sample N rays uniformly along a segment of length L of the line at right angles to the source-to-origin line, then $\delta\lambda = L/(N - 1)$, and the i^{th} ray corresponds to

$$\lambda_i = -L/2 + i\ \delta\lambda \tag{57}$$

Consequently, $g_j(\lambda')$ is found by interpolation from $g_{i'j}$ and $g_{(i' + 1)j}$ where,

$$i' = \lfloor(\lambda' + L/2)/\delta\lambda\rfloor \tag{58}$$

In practice, of course, detectors would not be arranged on a line passing through the scanned space. The geometric transformations from a more distant, linear detector array to positions on the line passing through the origin are fortunately trivial (see figure 3). Such an array of equally wide detectors positioned behind the object being scanned would have to move in synchrony with the source, so as to always remain perpendicular to the source-to-origin line. There is great interest in scanning schemes which can instead use a fixed array of detectors. One such arrangement will be discussed in the next section.

EVEN SAMPLING OF RAYS IN FAN ANGLE.

Even sampling of rays in fan angle α can be achieved easily using a set of equally wide detectors arrayed on a sector of a circle with center at the source position. Curiously equal spacing of samples in fan angle can also be achieved when these detectors are instead placed on a circle passing through the source, with center at the origin (see figure 7). This follows from the fact that the angle at the center is just twice the angle at the source, and so equal angular spacing of detectors when viewed from the origin corresponds to equal angular spacing of detectors when viewed from the source. Such an arrangement of detectors has an advantage in that the detectors could remain stationary during scanning if the potential mechanical conflict between source and detectors could be solved. In any case, it is natural here to let

$$\xi = \alpha \tag{59}$$

So,

$$D \cos \alpha \frac{\partial \alpha}{\partial \xi} = D \cos \alpha \tag{60}$$

Proceeding as in the previous section, we obtain (from equations 35 and 36),

$$f(r,\phi) = \frac{1}{4\pi^2} \int_0^{2\pi} g''(\alpha',\beta)(1/K^2)\, d\beta \tag{61}$$

$$g''(\alpha',\beta) = \lim_{\varepsilon \to 0} \int_{-\pi/2}^{+\pi/2} F_\varepsilon[\sin(\alpha - \alpha')]\, D \cos \alpha\, p(\alpha,\beta)\, d\alpha \tag{62}$$

where (by equation 25),

$$K^2 = r^2 + D^2 + 2rD \sin (\beta - \phi) \tag{63}$$

Next, we obtain the discrete approximation,

$$f(r,\phi) \approx \frac{1}{4\pi^2} \sum_j g_j(\alpha') \frac{1}{r^2 + D^2 + 2rD \sin (\beta_j - \phi)} \delta\beta_j \tag{64}$$

$$g_{i'j} = \sum_i F_{i'-i} \, D \cos \alpha_i \, P_{ij} \, \delta\alpha \tag{65}$$

where

$$\alpha' = \tan^{-1} \left[\frac{r \cos (\beta_j - \phi)}{D + r \sin (\beta_j - \phi)} \right] \tag{66}$$

Once again p_{ij} is simply the i^{th} ray sum in the j^{th} fan, while $\delta\alpha$ is the (fixed) angular interval between rays in the fan. The angular interval associated with a particular fan, $\delta\beta_j$, is as defined before (equation 40). The filter factors are:

$$F_k = - \frac{w_k}{\sin^2 (k \, \delta\alpha)} \qquad k \neq 0 \tag{67}$$

$$F_o = -\sum_{k \neq 0} F_k \tag{68}$$

Finally, one needs to detail the interpolation procedure for finding $g_j(\alpha')$ from the discrete set of values $g_{i'j}$. If N rays are sampled uniformly along an arc of angle A, then $\delta\alpha = A/(N - 1)$. The i^{th} ray then corresponds to

$$\alpha_i = -A/2 + i\ \delta\alpha \tag{69}$$

Consequently, $g_j(\alpha')$ is found by interpolation from $g_{i'j}$ and $g_{(i'+1)j}$ where

$$i' = \lfloor(\alpha' + A/2)/\delta\alpha\rfloor \tag{70}$$

This reconstruction method may be viewed as a pre-multiplication of the ray sums by $D \cos \alpha_i$, followed by convolution, with a final post-multiplication by $1/[r^2 + D^2 + 2rD \sin(\beta_j - \phi)]$. This is the second special case of the fan beam reconstruction problem which had been solved previously [5, 6].

Other fan-beam scanning geometries do <u>not</u> lead to such special case solutions however. Usually, a general linear operation is required. Fortunately, the method presented earlier allows one to treat <u>arbitrary</u> fan beam scanning geometries. We will study one in detail as an illustration.

A METHOD WITH UNIFORM SAMPLING DENSITY.

Both of the scanning schemes discussed so far sample areas near the origin less densely than they do areas near the edge of the region of reconstruction. This can be seen when it is remembered that the ray sampling density is the inverse of the Jacobian J [1] and that for fan beam scanning (equation 24),

$$J = \frac{\partial \beta}{\partial \eta} \; D \cos \alpha \; \frac{\partial \alpha}{\partial \xi} \tag{71}$$

Now for the first method (equation 38)

$$D \cos \alpha \left(\frac{\partial \alpha}{\partial \xi}\right) = \cos^3 \alpha \tag{72}$$

while for the second method (equation 66)

$$D \cos \alpha \left(\frac{\partial \alpha}{\partial \xi}\right) = D \cos \alpha \tag{73}$$

The result of this variation in sampling density is that reconstructions will have somewhat better resolution (particularly in the radial, as opposed to tangential, direction) in outlying regions. While this effect is not very pronounced for fans that are fairly narrow, it is still of interest to investigate schemes providing <u>uniform</u> sampling density. That is,

$$D \cos \alpha \left(\frac{\partial \alpha}{\partial \xi}\right) = 1 \tag{74}$$

If this equation is integrated one finds,

$$\xi = \ell = D \sin \alpha \tag{75}$$

This means that rays are spaced evenly in their perpendicular distance ℓ from the origin (see figure 8). No convenient arrangement of equally wide detectors will provide for sampling of the fan in this fashion, but clearly detectors of <u>varying</u> width arranged on either a straight or curved line can be used. Their width will increase with distance from the central detector.

Now note that

$$\sin(\alpha - \alpha') = \sin\alpha \cos\alpha' - \sin\alpha' \cos\alpha \tag{76}$$

$$\sin(\alpha - \alpha') = \cos\alpha'(\tan\alpha - \tan\alpha')\cos\alpha \tag{77}$$

$$\sin(\alpha - \alpha') = \frac{\sqrt{D^2 - \xi'^2}}{D}\left[\frac{\xi}{\sqrt{D^2 - \xi^2}} - \frac{\xi'}{\sqrt{D^2 - \xi'^2}}\right]\frac{\sqrt{D^2 - \xi^2}}{D} \tag{78}$$

where $\xi' = D \sin \alpha'$. Proceeding as before (using equations 35 and 36), we get:

$$f(r,\phi) = \frac{1}{4\pi^2}\int_0^{2\pi} g''(\ell',\beta)\,\frac{D^2}{[D + r\sin(\beta - \phi)]^2}\,d\beta \tag{71}$$

$$g''(\ell',\beta) = \lim_{\varepsilon \to 0}\int_{-D}^{+D} F_\varepsilon\left(\frac{\ell}{\sqrt{D^2 - \ell^2}} - \frac{\ell'}{\sqrt{D^2 - \ell'^2}}\right)\frac{1}{(D^2 - \ell^2)}\,p(\ell,\beta)\,d\ell \tag{80}$$

The following identity was used for the outer integral,

$$\frac{1}{K^2}\cdot\frac{D^4}{D^2 - \ell'^2} = \frac{D^4}{K^2 D^2 \cos^2\alpha'} = \frac{D^2}{[D + r\sin(\beta - \phi)]^2} \tag{81}$$

Note that here ℓ' is the value of ℓ for the ray inclined α' to the source-to-origin line (see figure 8),

$$\ell' = D \sin \alpha' \tag{82}$$

(This differs from the parallel ray scanning schemes presented in the previous paper [1]).

Once again, a discrete approximation is required,

$$f(r,\phi) \simeq \frac{1}{4\pi^2} \sum_j g_j(\ell') \frac{D^2}{[D + r \sin (\beta_j - \phi)]^2} \delta\beta_j \tag{83}$$

$$g_{i'j} = \sum_i F_{i'i} \frac{1}{D^2 - \ell_i^2} p_{ij} \, \delta\ell \tag{84}$$

where

$$\ell' = D \sin \alpha' = \frac{Dr \cos(\beta_j - \phi)}{\sqrt{r^2 + D^2 + 2rD \sin(\beta_j - \phi)}} \tag{85}$$

The filter factors are

$$F_{i'i} = - \frac{w_K}{\left[\frac{\ell_i}{\sqrt{D^2 - \ell_i^2}} - \frac{\ell_{i'}}{\sqrt{D^2 - \ell_{i'}^2}}\right]^2} \quad \text{for } i \neq i' \tag{86}$$

$$F_{i'i'} = - \sum_{i \neq i'} F_{i'i} \tag{87}$$

In this case, then, as for most scanning schemes, a general linear operator rather than a modified convolution must be used. The methods presented here permit the derivation of algorithms to deal with these problems. Note by the way that here the factors of sin $(\alpha - \alpha')$ were split up in a similar fashion to how this had been done for the first two examples. This is not strictly necessary, since all three components can be accomodated as part of the filter function $F_\varepsilon(\ell,\ell')$ or $F_{i'i}$ if so desired.

ANOTHER METHOD.

The existence of an elegant method for reconstruction from parallel-beam data (equations 7 & 8 or 9 & 10) which uses derivatives of projection data and does not depend on arbitrary filter coefficients leads one to search for a similar expression for fan-beam reconstruction. Starting from the general form (equation 4) does not seem to lead to such a result. Instead one may apply partial integration to the form of the inner integral shown in equation 36,

$$g''(\alpha',n) = \lim_{\varepsilon\to 0} \int_{-\pi/2}^{\pi/2} F_\varepsilon[\sin(\alpha - \alpha')]\, p(\alpha,\beta)\, D \cos\alpha\, d\alpha \tag{88}$$

If one lets $\sin(\delta) = \varepsilon$, then

$$g''(\alpha',n) = \lim_{\varepsilon\to 0} \Big[\int_{-\pi/2}^{\alpha'-\delta} - \frac{1}{\sin^2(\alpha - \alpha')}\, p(\alpha,\beta)\, D \cos\alpha\, d\alpha + \frac{1}{\varepsilon^2} \int_{\alpha'-\delta}^{\alpha'+\delta} p(\alpha,\beta)\, D \cos\alpha\, d\alpha + \int_{\alpha'+\delta}^{+\pi/2} - \frac{1}{\sin^2(\alpha - \alpha')}\, p(\alpha,\beta)\, D \cos\alpha\, d\alpha \Big] \tag{89}$$

Or,

$$g''(\alpha',\eta) = \lim_{\varepsilon\to 0}\left[\int_{-\pi/2}^{\alpha'-\delta} - \frac{\cos(\alpha-\alpha')}{\sin^2(\alpha-\alpha')}\, p(\alpha,\beta)\, \frac{D\cos\alpha}{\cos(\alpha-\alpha')}\, d\alpha\right] + \tag{90}$$

$$\frac{2\delta}{\varepsilon^2}\, p(\alpha',\beta) D\cos\alpha' +$$

$$\lim_{\varepsilon\to 0}\left[\int_{\alpha'+\delta}^{+\pi/2} - \frac{\cos(\alpha-\alpha')}{\sin^2(\alpha-\alpha')}\, p(\alpha,\beta)\frac{D\cos\alpha}{\cos(\alpha-\alpha')}\, d\alpha\right]$$

That is,

$$\left[\frac{D}{\sin(\alpha-\alpha')}\, p(\alpha,\beta)\frac{\cos\alpha}{\cos(\alpha-\alpha')}\right]_{-\pi/2}^{\alpha'-\delta} -$$

$$\int_{-\pi/2}^{\alpha'-\delta} \frac{D}{\sin(\alpha-\alpha')}\, \frac{\partial}{\partial\alpha}\left[p(\alpha,\beta)\, \frac{\cos\alpha}{\cos(\alpha-\alpha')}\right] d\alpha +$$

$$\frac{2D}{\delta}\, p(\alpha',\beta)\cos\alpha' + \left.\frac{D}{\sin(\alpha-\alpha')}\, p(\alpha,\beta)\, \frac{\cos\alpha}{\cos(\alpha-\alpha')}\right]_{a'+\delta}^{+\pi/2} -$$

$$\int_{\alpha'+\delta}^{+\pi/2} \frac{D}{\sin(\alpha-\alpha')}\, \frac{\partial}{\partial\alpha}\left[p(\alpha,\beta)\frac{\cos\alpha}{\cos(\alpha-\alpha')}\right] d\alpha \tag{91}$$

As $\delta \to 0$ this becomes simply,

$$- \int_{-\pi/2}^{+\pi/2} \frac{D}{\sin(\alpha - \alpha')} \frac{\partial}{\partial\alpha} [p(\alpha,\beta) \frac{\cos\alpha}{\cos(\alpha - \alpha')}] \, d\alpha \tag{92}$$

To summarize,

$$f(r,\phi) = \frac{1}{4\pi^2} \int_{o}^{2\pi} g(\alpha',\beta) \frac{D}{K^2(r,\phi,\beta)} \, d\beta \tag{93}$$

$$g(\alpha',\beta) = - \int_{-\pi/2}^{\pi/2} \frac{1}{\sin(\alpha - \alpha')} \frac{\partial}{\partial\alpha} [p(\alpha,\beta)\frac{\cos\alpha}{\cos(\alpha - \alpha')}] \, d\alpha \tag{94}$$

Where, as before,

$$K^2 = r^2 + D^2 + 2r\,D \sin(\beta - \phi) \tag{95}$$

$$\tan\alpha' = \frac{r\cos(\beta - \phi)}{[D + r\sin(\beta - \phi)]} \tag{96}$$

The discrete approximation is

$$f(r,\phi) \simeq \frac{1}{4\pi^2} \sum_{j} g_j(\alpha') \frac{D}{K^2(r,\phi,\beta_j)} \delta\beta_j \tag{97}$$

$$g_{i'j} = - \sum_{i} \frac{1}{\sin(\alpha_i - \alpha'_{i'}} [p(d_i,\beta) \frac{\cos(\alpha'_i)}{\cos(\alpha'_i - \alpha'_{i'})} - p(\alpha_{i-1},\beta) \quad \frac{\cos(\alpha'_{i-1})}{\cos(\alpha'_{i-1} - \alpha'_{i'})}] \tag{98}$$

Here α_i corresponds to the left edge of the i^{th} detector, while α_{i+1} marks its right edge (see figure 9). The ray-sum seen by the i^{th} detector is p_{ij} and its center is at α_i'.

CONCLUSION AND SUMMARY.

The formulas for reconstruction from ray sums obtained by arbitrary sampling schemes were specialized to a system utilizing fan beams originating from sources on the circumference of a circle. It was found that one need not calculate the contribution of each ray sum to each point explicitly, but that the calculation does involve the application of a general linear operator. In special cases, this linear operator becomes space invariant by a manipulation of the integrals, and the superposition integral simplifies into a convolution. Two examples of this were shown, both corresponding to previously known solutions to the fan beam reconstruction problem for particular ray collection geometries.

To illustrate the utility of the new method, however, a third case was considered where the simplification does not occur. Previous techniques for finding reconstruction methods based on Fourier transforms cannot deal with it. Details of an algorithm were developed. The utility of the new methods for finding algorithms for arbitrary fan beam scanning schemes is therefore apparent. The introduction of uniform scanning coordinates in particular is of great importance in finding reconstruction methods for the general case.

ACKNOWLEDGMENTS

Figures are by Karen Prendergast.

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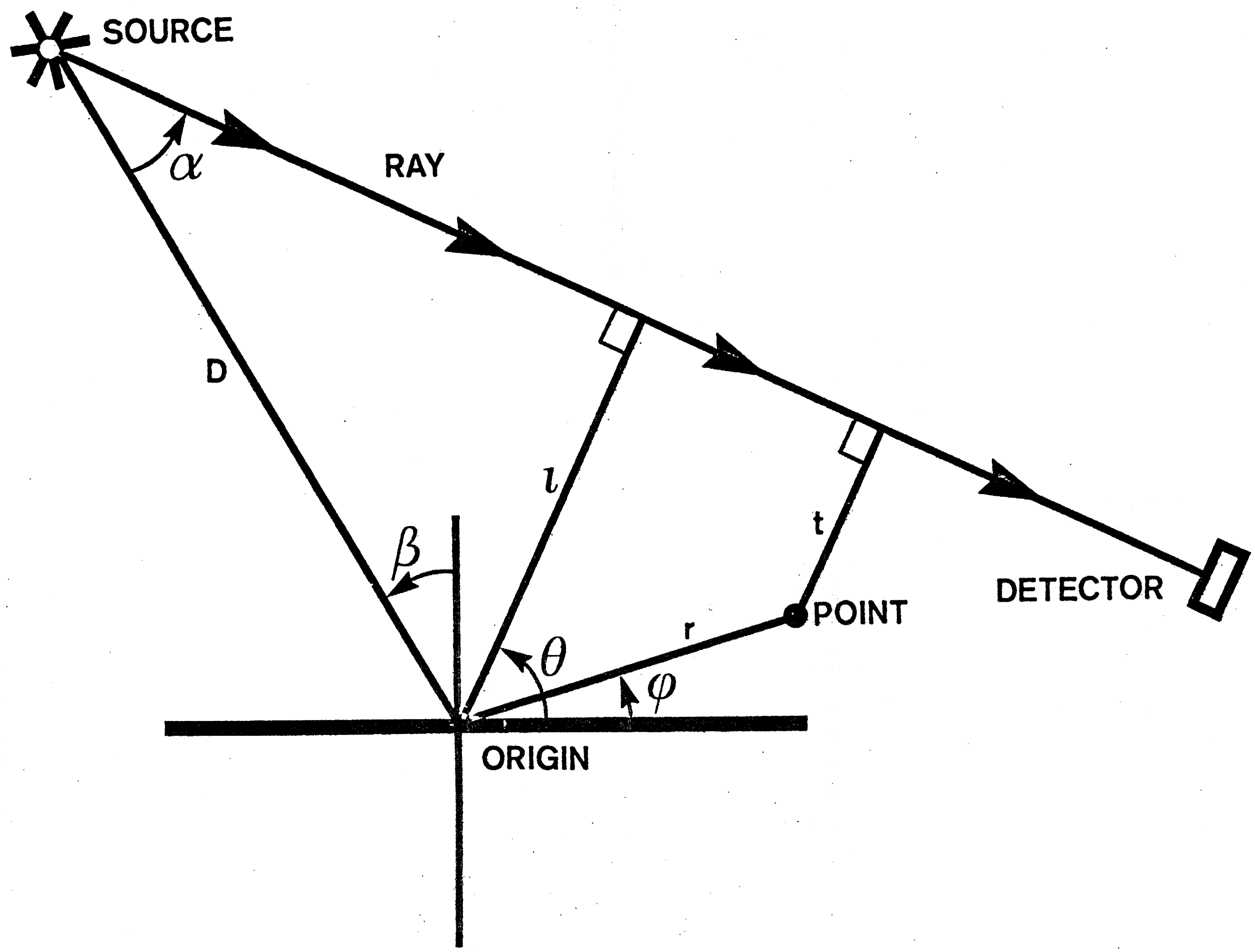


FIGURE I

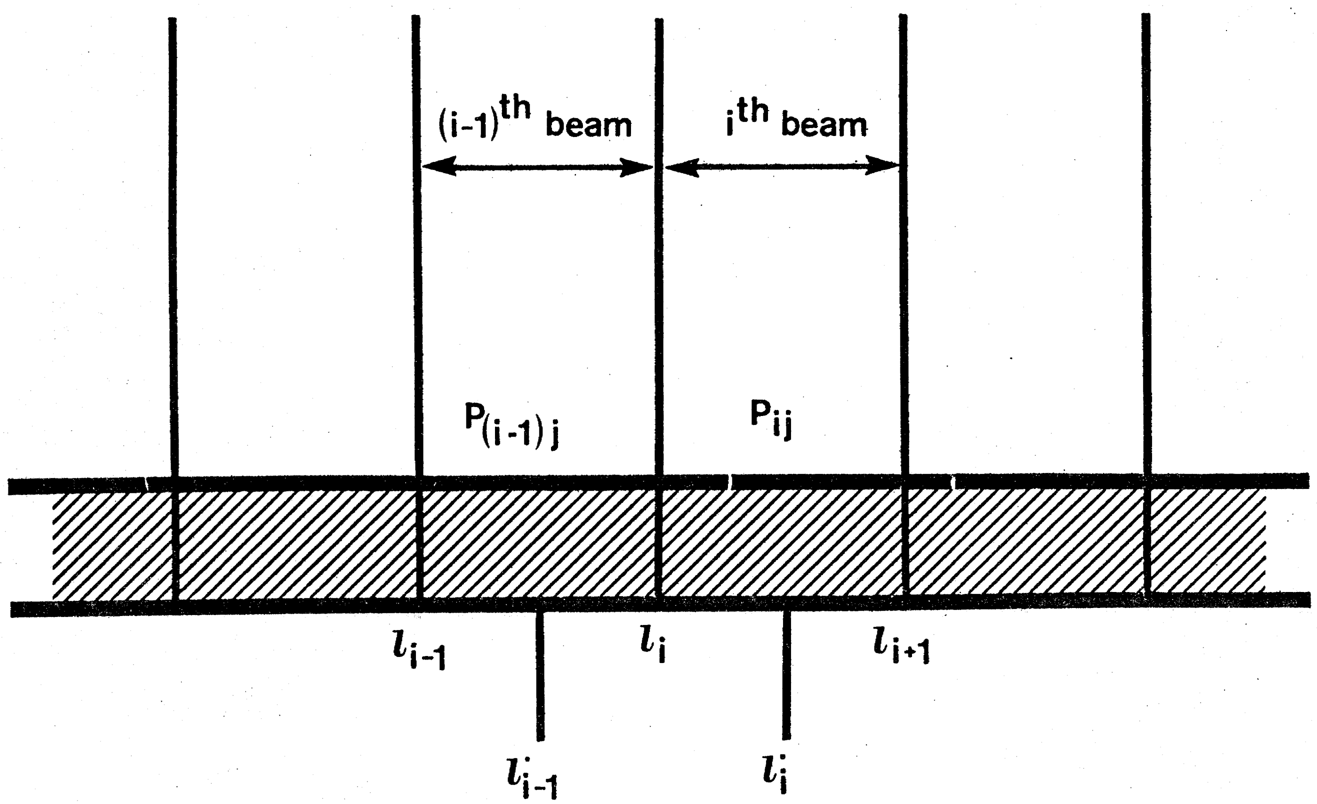


FIGURE 2

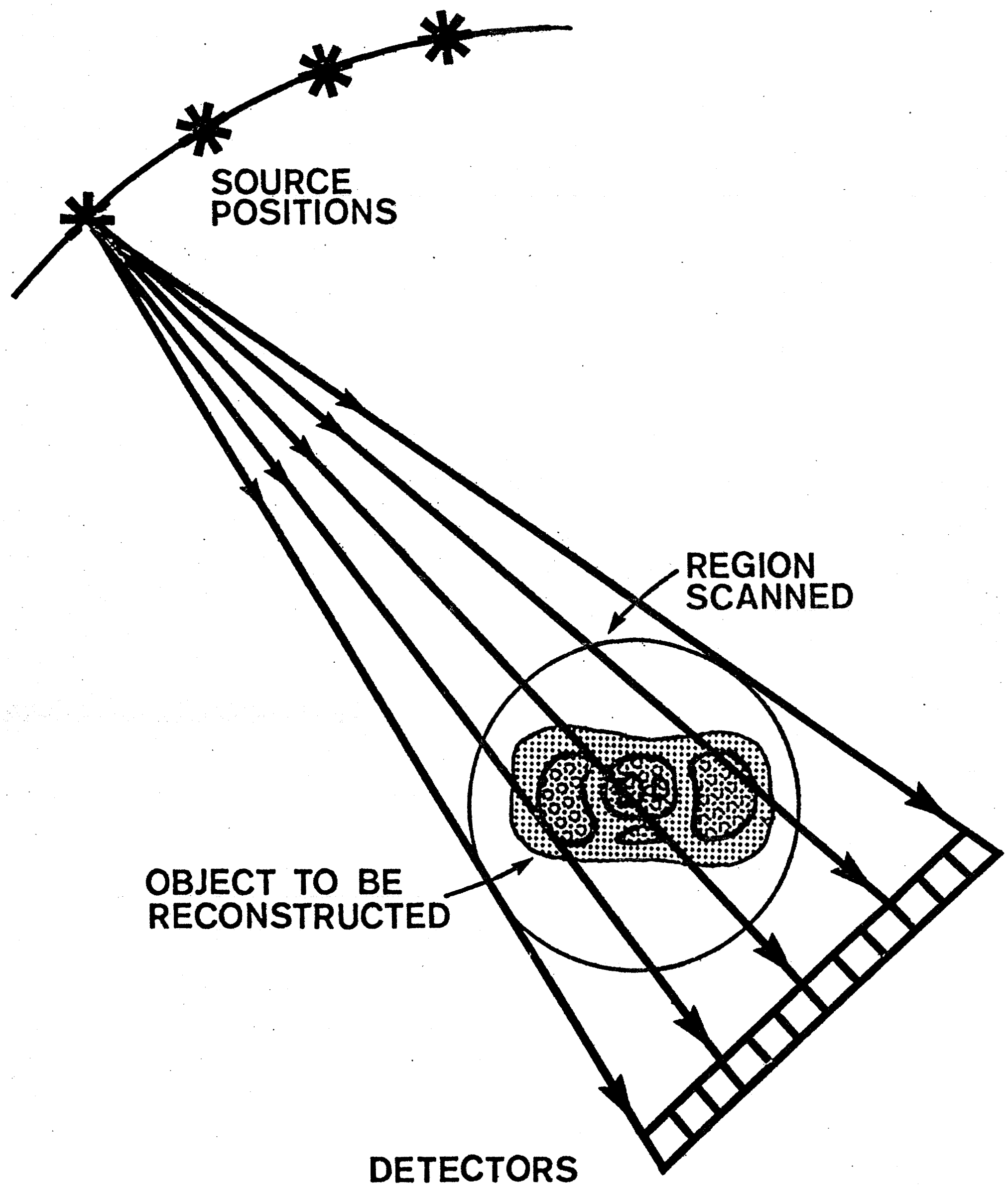
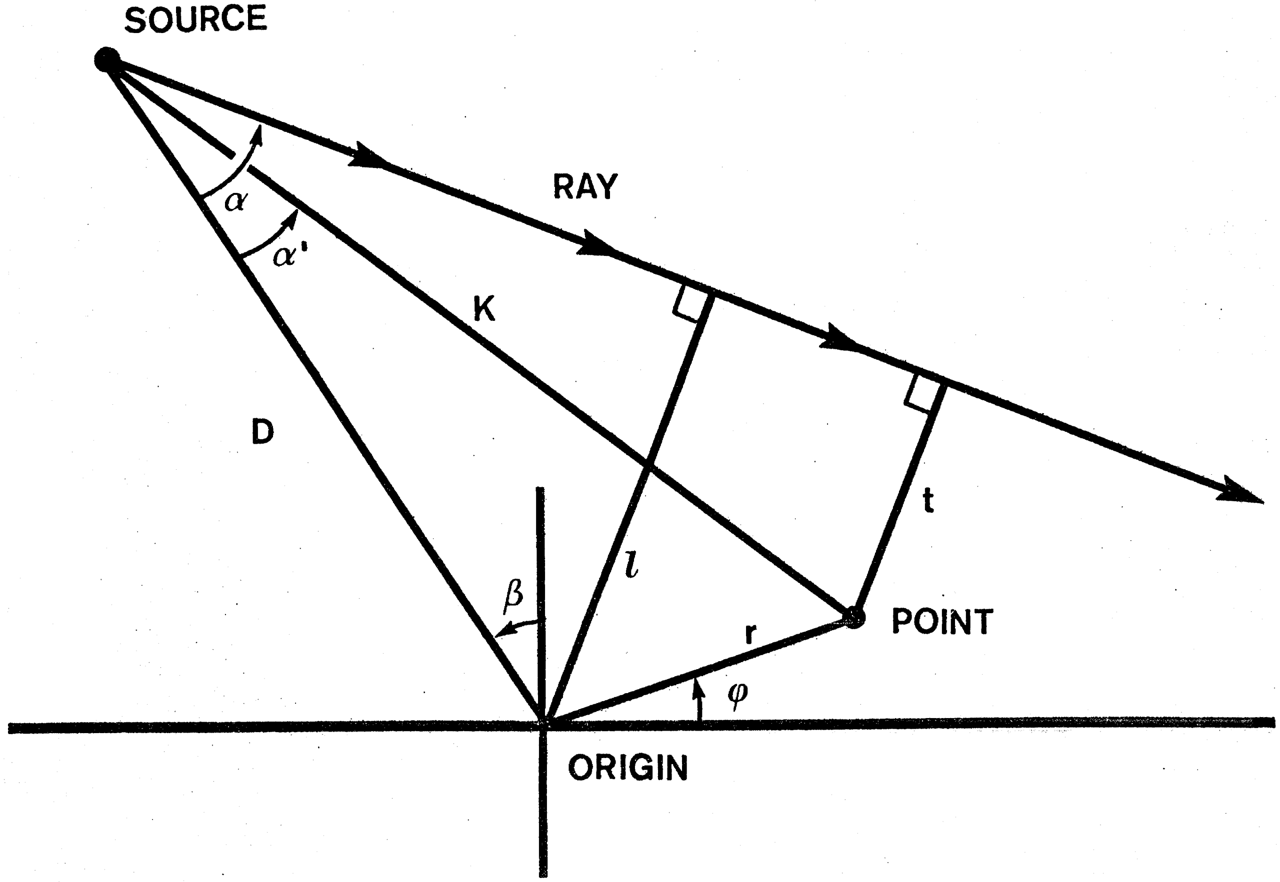

SOURCE
POSITIONS
REGION
SCANNED
OBJECT TO BE
RECONSTRUCTED
DETECTORS

FIGURE 3



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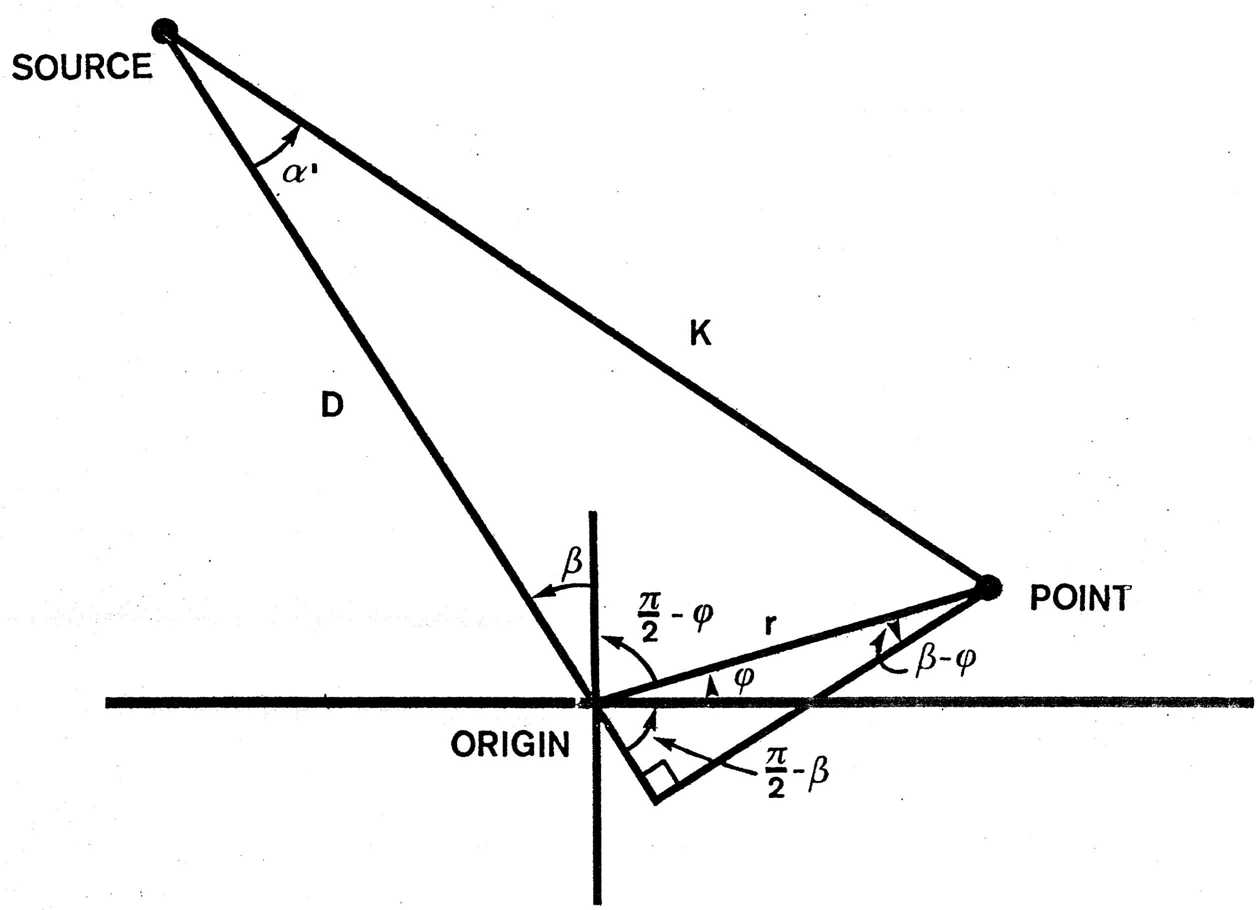

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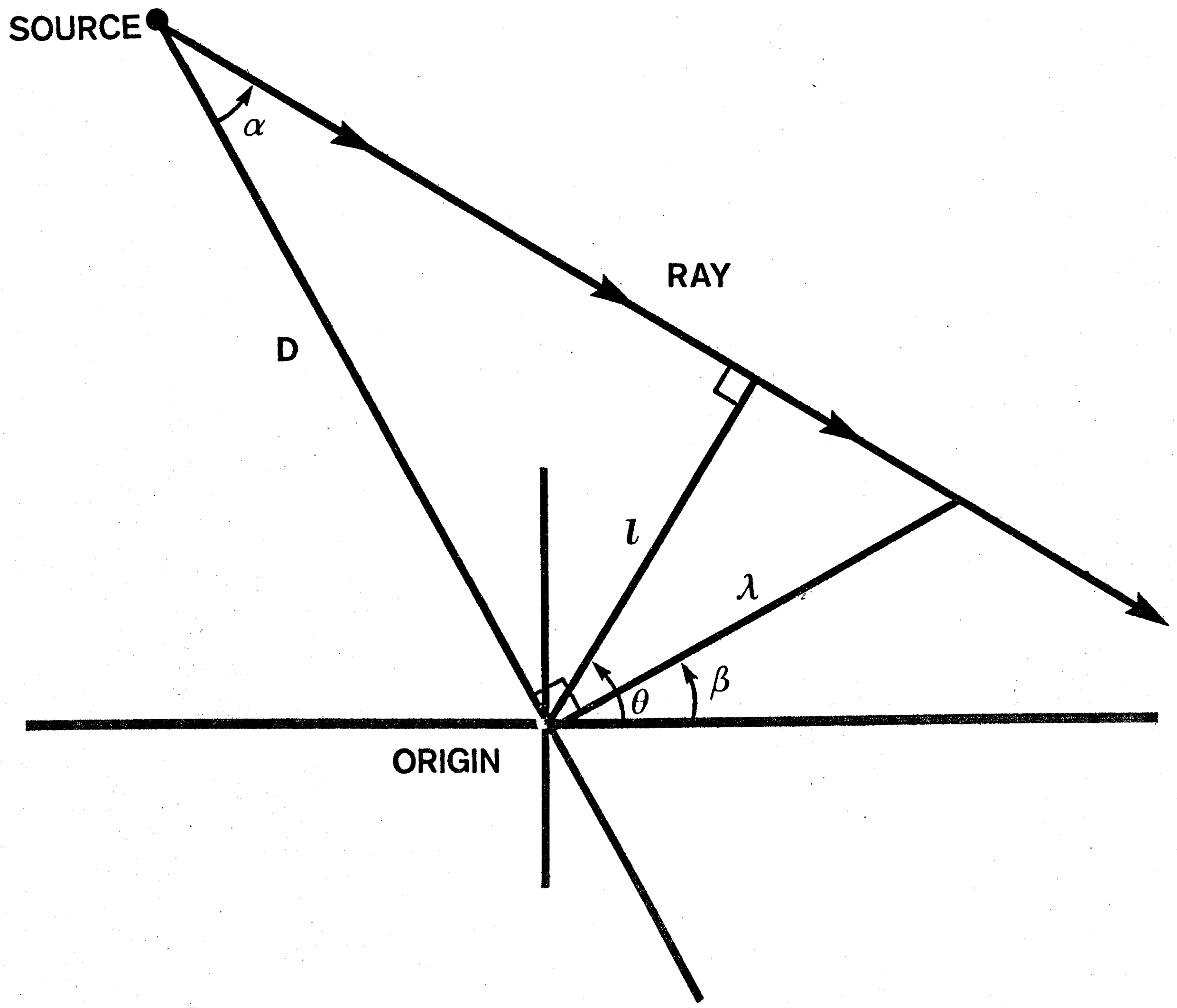
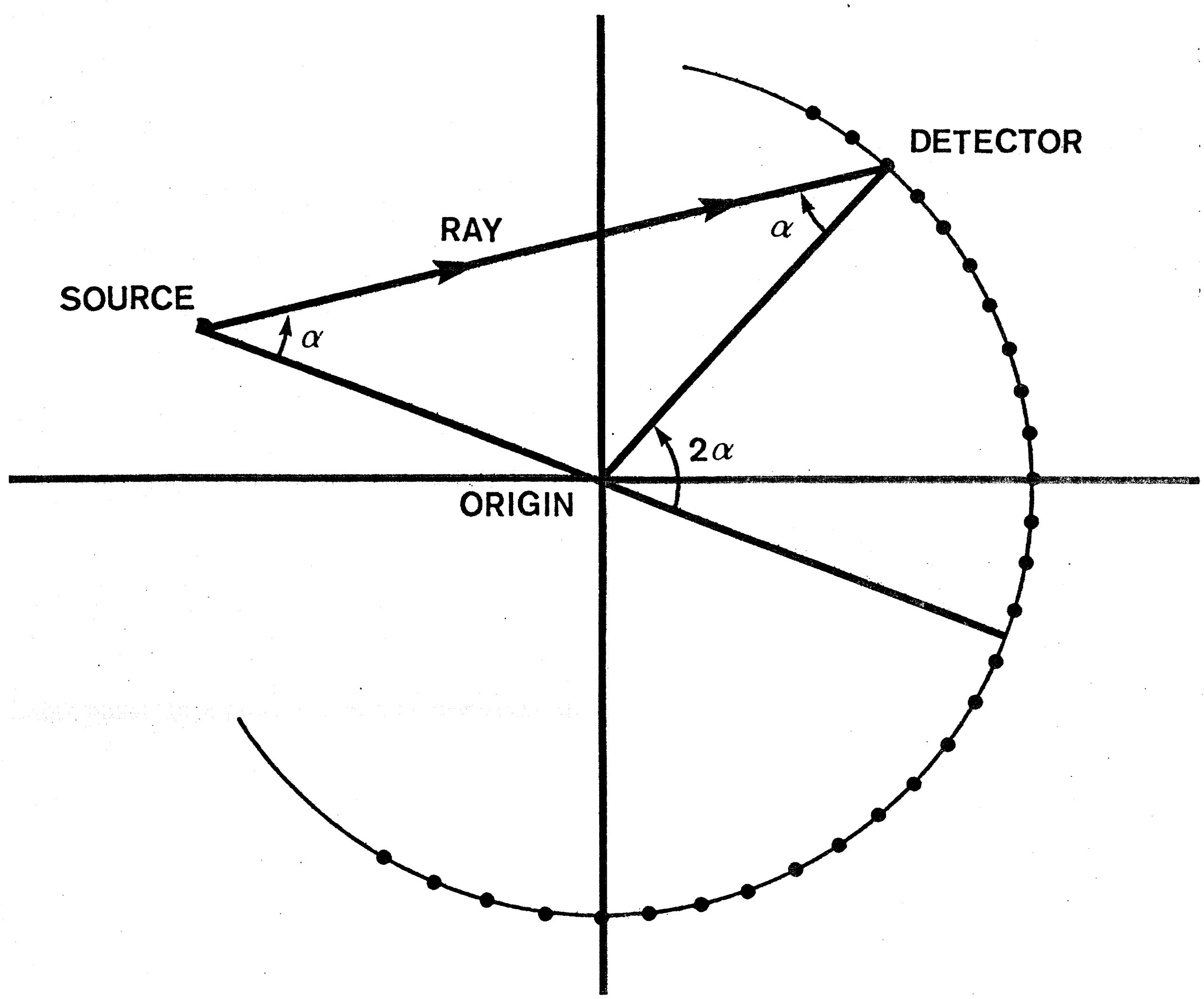


FIGURE 6



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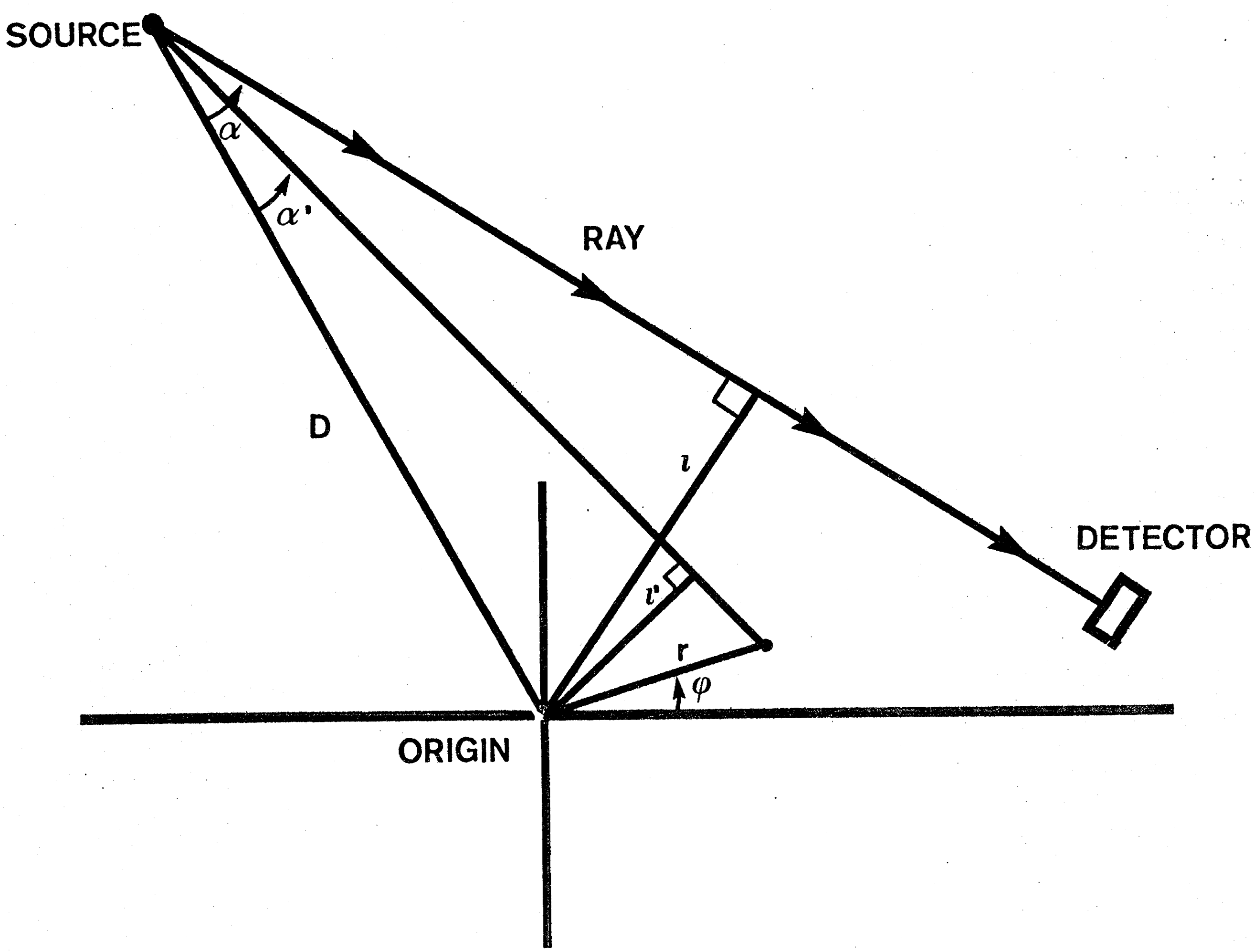

FIGURE 8

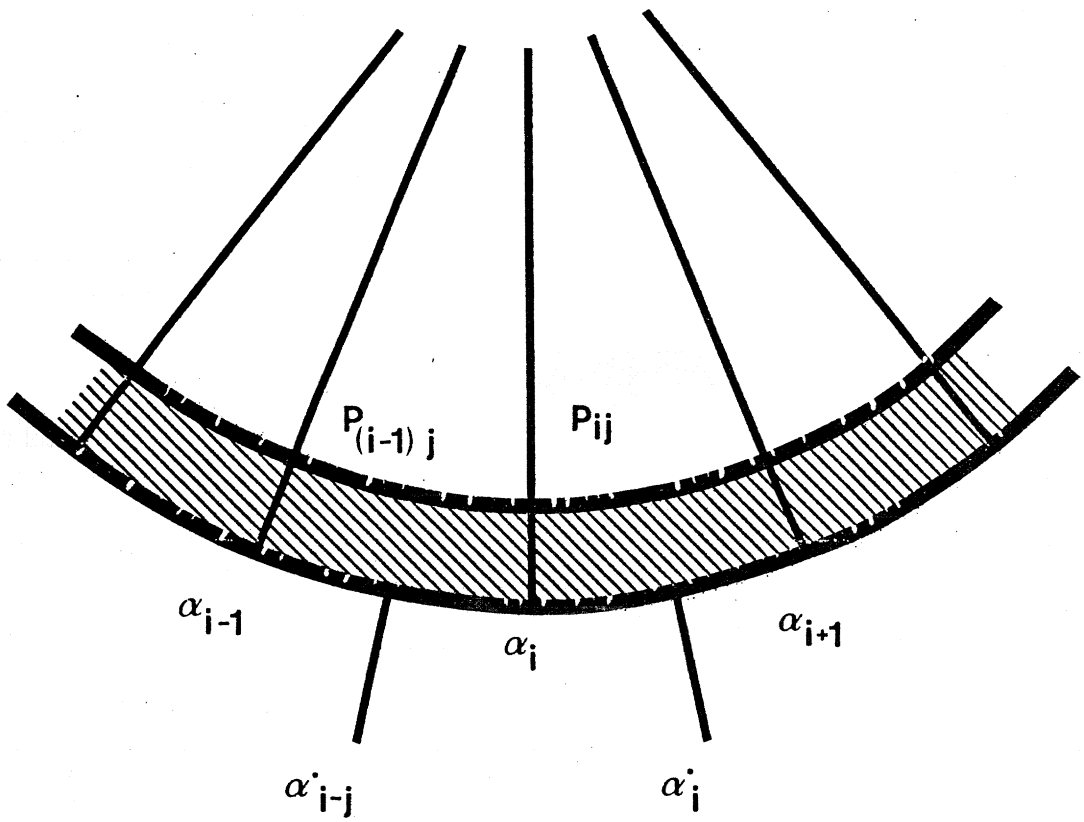

FIGURE 9