RECOMP II USERS' PROGRAM NO. 1112

PROGRAM TITLE: BESSEL FUNCTION - $J_n(x)$ - SUBROUTINE (FLOATING POINTS)

PROGRAM CLASSIFICATION: Subroutine

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PURPOSE: To provide a subroutine to compute for a given argument x, all significant orders of Bessel functions of the first kind $J_n(x)$.

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BESSEL FUNCTION - $J_n(x)$ - SUBROUTINE (FLOATING POINTS)

1. Purpose

To provide a subroutine to compute for a given argument x, all significant orders of Bessel functions of the first kind $J_n(x)$.

2. Restrictions

The argument x must be in the range $0 < x < 207.0631$.

3. Method

3.1 From the value of the argument x, a starting order is determined from the relationship:

$$n_s = 7 + (65.518x + 1.1765x^2)^{\frac{1}{2}} \quad \text{(truncated to a multiple of 4)}$$

The value n_s, which is given in Table I as a function of x, is the total number of orders generated for argument x, and may be defined as the smallest multiple of 4 that satisfies the inequality:

$$J_n(x) < 1 \times 10^{-11} \quad \text{for all } n \geqslant n_s$$

Arbitrary values of 2 and 0, respectively, are assigned to $J_{n_s}(x)$ and $J_{n_s+1}(x)$. The subroutine then computes all orders down to and including $J_o(x)$ successively from the recurrence relation:

$$J_{n-1}(x) = \left|\frac{2n}{x}\right| J_n(x) - J_{n+1}(x)$$

The resulting table of n_s unnormalized Bessel functions, from $J_o(x)$ to $J_{n_s-1}(x)$, inclusive, is then corrected by dividing each entry by a normalization constant N determined from the unnormalized values in the following way:

$$N = J_o(x) + 2 \sum_{i=1}^{\infty} J_{2n}(x)$$

with the summation terminating with the term $J_{n_s}(x)$. Actually, in the subroutine, the summation is accumulated as the unnormalized Bessel functions are generated.

3.2 Accuracy

The accuracy of the $J_n(x)$ functions obtained with this subroutine depends upon the portion of the particular Bessel function in question.

3.2.1 Monotonic Region ($x < n$)

Since in this region the $J_n(x)$ functions contain no zeros (except at the origin), the concept of relative error is to be preferred. The relative error in this region is smallest at the order nearest x, and gradually rises smoothly, for successively higher orders, until an order about 5 or 6 below the highest computed order (n_s-1) is reached. From this point to the highest computed order, the relative error rapidly increases and is largest for the highest order.

Typical errors for this region are as follows:

RELATIVE ERROR FOR VARIOUS ORDERS ($x < n$)

x	n	relative error	good* decimal places	n	relative error	good* decimal places	highest n	relative error	good* decimal places
1	1	0.1×10^{-10}	11	5	0×10^{-10}	14	11	0.03×10^{-4}	16
6	6	3.7 "	10	17	13	15	23	2.6	15
12	12	4.6 "	10	29	16	18	35	8.8	16
18	18	6.4 "	10	37	22	17	43	21.0	15
24	24	8.4 "	10	45	37	17	51	36.0	15

*Last place agreement within two units.

Lack of adequate reference tables prevents an extension of the foregoing table to higher values of x.

3.2.2 Oscillating region ($x > n$)

In this region, the $J_n(x)$ functions are periodic with distances between zeros of 3-9 units in x depending upon the order. Thus absolute error appears to be a more useful measure in this region.

The absolute error of the $J_n(x)$ functions as computed by this subroutine is highest in the vicinity of the zeros, and lowest in the vicinity of the maximums and minimums. An analysis of the errors in the vicinities of the zeros reveals that the average error obtained may be expressed

$$|\text{average absolute error}| \sim (3 + 0.21x)10^{-10} \text{ in the vicinity of the zeros}$$

The errors obtained in the vicinity of zeros have considerable scatter, and maximum errors may be twice the mean error approximated above. In addition, when the derivatives of the function in the vicinity of the zero is positive, the calculated values given by this subroutine are nearly always greater than the true values and vice versa. Although the above expression for the zero-vicinity error was derived principally from $J_0(x)$ and $J_{20}(x)$ data, it appears to be applicable for all orders given by this subroutine.

The average error in the vicinity of the maxima and minima of the $J_n(x)$ functions appears to be more or less independent of both x and the order n. Typical absolute errors have been observed to be about 3×10^{-11} with maximum errors less than 6×10^{-11}.

4. Usage

4.1 The argument must be a floating point number in the A and R registers.

4.2 Calling sequence

α)	FCA argument
	TRA L_o
α + 1)	PZE $NMAX_8$
	PZE $NREQ_8$
α + 2)	Error return
	Normal return

4.3 Explanation of symbols

4.3.1 $NMAX_8$ is an octal interger (at binary point 18) that serves to specify the storage allocation of the $J_n(x)$ table. $NMAX_8$ normally has the value 400_8, which provides for storing the maximum of 256_{10} orders (from J_o to J_{255}, inclusive) designed for this subroutine. J_o is always stored at location L_o + 0300_8, J_1 at location L_o + 0302_8,...,J_{255} at location L_o + 1276_8.

If a user knows that his arguments will never exceed 100, for example, then reference to Table I shows that 140_{10} orders will be generated (from J_o to $J_{139_{10}}$, inclusive), or in octal 214_8 orders will be generated. Thus by setting $NMAX_8$ = 214_8, twice this number (2 locations are required per order) or 430_8 locations are reserved for the J_n table. Two objectives are therefore accomplished:

a. Memory locations starting with location L_o + 0730_8 are unused and are thus available for other purposes.

b. Arguments in excess of 101.4945 (from Table I will cause an error return.

4.3.2 This subroutine normally computes all significant orders for a given argument x, with the total number of orders computed given in Table I as a function of x. For this normal type of operation, $NREQ_8$, an octal integer (at binary point 38) should be set to zero.

But if, for example, a type of problem arises where say the first 20_{10} orders are always required, regardless of the value of x, then NREQ may be set to 0024_8. The subroutine calculates the normal starting order, and then raises it, if necessary, in jumps of 4 to provide a starting order of from 4-7 higher orders than that specified by NREQ. This assures that all orders up to and including that specified by NREQ will be obtained to the full accuracy of the subroutine.

4.4.1 Error return is implemented when the argument is zero, negative, or exceeds the limits imposed by NMAX (see example in 4.3.1.b). The error return is made with the argument x in the A and R registers.

4.4.2 Normal return is made with the address of the highest order computed in the accumulator at binary point 38. The highest order itself (n_s-1) is displayed as a decimal integer.

4.5 The subroutine is relocatable. Relocation subroutine RUG 1075 is included on the tape.

4.6 The subroutine destroys the L and V loops.

4.7 Options available are discussed in 4.3.1 and 4.3.2.

5. <u>Coding Information</u>

5.1 The subroutine occupies three tracks or 192_{10} words. The J_n table always starts with J_o at location L_o + 0300_8 and extends up to a maximum value of J_{255} at L_o + 1276_8.

5.2 Unused locations. A total of 13 locations are unused as follows:

L_o +	0003
	0006
	0021
	0022
	0101
	0111
	0121
	0127
	0141
	0151
	0215
	0223
	0277

5.3 Locations used for temporary storage total 10 as follows:

L_o +	0010
	0011
	0013
	0023
	0075
	0076
	0077
	0150
	0251
	0252

5.4 Constants (floating point)

1.1765	L_o +	0136
2		0216
65.518		0226
7		0270

Constants (fixed point)

10 at 39	L_o +	0212
0		0214

5.5 Execution time. The execution time for the subroutine depends upon the number of orders generated and is approximately as follows:

$t = n_s/10$ where n_s is the number of orders generated

6. Checkout

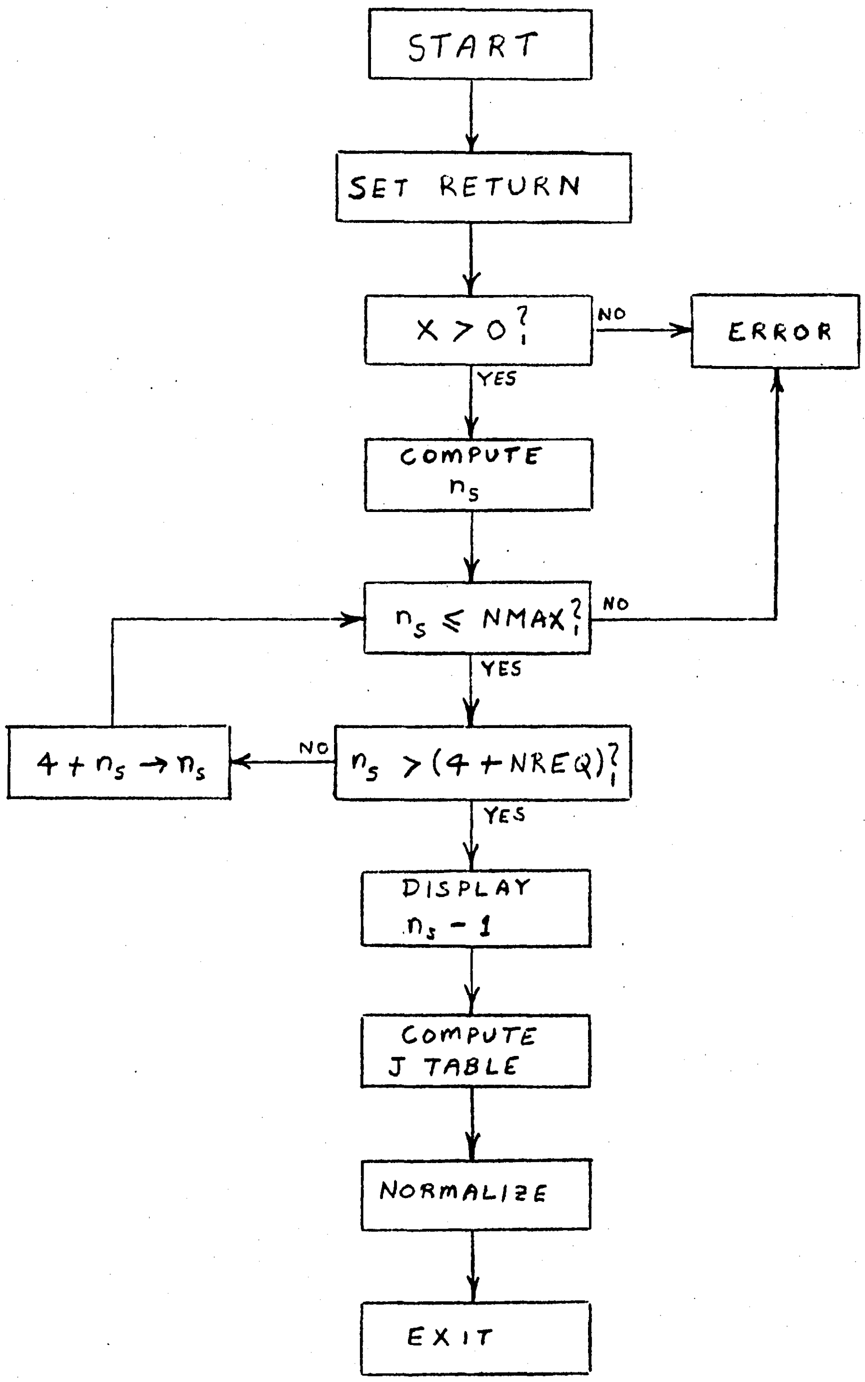
To check out this subroutine, numerous comparisons were made with the classic tables of Bessel functions. The accuracy figures given in 3.2 stemmed from an analysis of these comparisons. Reference tables include:

a. Meissel's 18-decimal table of $J_n(x)$ at unit interval in x up to x = 24 as reprinted in Gray, Mathews, and MacRobert, "A Treatise on Bessel Functions," second edition, London, Macmillan, 1922.

b. Harvard Computation Laboratory. Tables of the Bessel functions of the first kind, 12 vol. 1947-1951. Gives, for x less than 100, 18 decimal values for n = 0 to n = 3, inclusive, and 10 decimal values for all higher orders. Interval in x ranges from 0.001 to 0.1.

TABLE I

x	n_g	x	n_g	x	n_g
0.	8	69.3458	108	159.5463	208
0.3790	12	72.8845	112	163.1939	212
1.2100	16	76.4336	116	166.8431	216
2.4699	20	79.9920	120	170.4936	220
4.1080	24	83.5587	124	174.1456	224
6.0695	28	87.1332	128	177.7987	228
8.3018	32	90.7145	132	181.4532	232
10.7580	36	94.3022	136	185.1087	236
13.3980	40	97.8957	140	188.7654	240
16.1889	44	101.4945	144	192.4230	244
19.1037	48	105.0982	148	196.0817	248
22.1207	52	108.7063	152	199.7413	252
25.2226	56	112.3186	156	203.4018	256
28.3952	60	115.9348	160	207.0631	
31.6273	64	119.5545	164		
34.9097	68	123.1774	168		
38.2348	72	126.8035	172		
41.5966	76	130.4324	176		
44.9899	80	134.0639	180		
48.4106	84	137.6980	184		
51.8551	88	141.3343	188		
55.3205	92	144.9729	192		
58.8042	96	148.6135	196		
62.3042	100	152.2560	200		
65.8186	104	155.9003	204		
69.3458		159.5463			



FLOW CHART FOR $J_n(x)$

CODING SHEETS

```
0000.0 +15.7764.0  SAX
       +57.0012.0  TRA

0001.0 +00.7774.0  CLA
       +22.7772.0  DIV

0002.0 +50.0115.0  TZE
       +57.0113.0  TRA

0003.0 -00.0000.0  ---
       -00.0000.0  ---

0004.0 +07.7772.0  FMP
       +57.0172.0  TRA

0005.0 +30.0010.0  FCA
       +57.0016.0  TRA

0006.0 -00.0000.0  ---
       -00.0000.0  ---

0007.0 +60.7770.0  STO
       +57.0025.0  TRA


0010.0 +00.0000.0  ---
       -00.0000.0  ---

0011.0 +00.0000.0  ---
       -00.0000.0  ---

0012.0 +01.0015.0  ADD
       +57.0126.0  TRA

0013.0 +00.0000.0  ---
       -00.0000.0  ---

0014.0 +05.7776.0  FDV
       +57.0103.0  TRA

0015.0 +30.7760.0  FCA
       +57.0001.1  TRA

0016.0 +43.0000.0  XAR
       +57.0225.0  TRA

0017.0 +35.7762.0  FST
       +57.0031.0  TRA

0020.0 +35.7772.0  FST
       +57.0034.0  TRA

0021.0 -00.0000.0  ---
       -00.0000.0  ---

0022.0 -00.0000.0  ---
       -00.0000.0  ---

0023.0 +00.0000.0  ---
       -00.0000.0  ---

0024.0 +35.7766.0  FST
       +57.0040.0  TRA

0025.0 +30.7776.0  FCA
       +57.0142.0  TRA

0026.0 +06.7774.0  FSB
       +57.0055.0  TRA

0027.0 +00.0000.0  ---
       -00.0000.1  ---


0030.0 +30.7776.0  FCA
       +57.0041.0  TRA

0031.0 +30.7760.0  FCA
       +57.0144.0  TRA

0032.0 +30.7776.0  FCA
       +57.0045.0  TRA

0033.0 +35.7760.0  FST
       +57.7772.1  TRA

0034.0 +30.7766.0  FCA
       +57.0044.0  TRA

0035.0 +03.0037.0  SUB
       +57.0146.0  TRA

0036.0 +43.0000.0  XAR
       +57.0046.0  TRA

0037.0 +57.0001.0  ---
       -20.7535.0  ---
```

```
0040.0 +30.7776.0  FCA
       +57.0050.0  TRA

0041.0 +35.7774.0  FST
       +57.0052.0  TRA

0042.0 +06.7774.0  FSB
       +57.0071.0  TRA

0043.0 +35.0010.0  FST
       +57.0057.0  TRA

0044.0 +35.7776.0  FST
       +57.0054.0  TRA

0045.0 +35.7774.0  FST
       +57.0062.0  TRA

0046.0 +66.0250.0  CTV
       +57.0065.0  TRA

0047.0 +03.0051.0  SUB
       +57.0160.0  TRA


0050.0 +35.7774.0  FST
       +57.0063.0  TRA

0051.0 +30.7607.1  ---
       +77.7777.1  ---

0052.0 +30.7762.0  FCA
       +57.0064.0  TRA

0053.0 +60.7773.0  STO
       +57.0001.0  TRA

0054.0 +07.7772.0  FMP
       +57.0042.0  TRA

0055.0 +35.7760.0  FST
       +57.0066.0  TRA

0056.0 +00.7770.0  CLA
       +57.0166.0  TRA

0057.0 +30.7772.0  FCA
       +57.0073.0  TRA
```

```
0060.0 +43.0000.0  XAR
       +57.0001.0  TRA

0061.0 +36.7773.1  DIS
       +57.0221.0  TRA

0062.0 +30.7764.0  FCA
       +57.0072.0  TRA

0063.0 +30.7772.0  FCA
       +57.0074.0  TRA

0064.0 +35.7776.0  FST
       +57.0174.0  TRA

0065.0 +06.7760.0  FSB
       +57.0153.0  TRA

0066.0 +04.0010.0  FAD
       +57.0143.0  TRA

0067.0 +35.7764.0  FST
       +57.0102.0  TRA


0070.0 +00.0004.0  ---
       -00.0010.0  ---

0071.0 +35.7764.0  FST
       +57.0105.0  TRA

0072.0 +35.7776.0  FST
       +57.0004.0  TRA

0073.0 +06.0076.0  FSB
       +57.0120.0  TRA

0074.0 +06.0076.0  FSB
       +57.0020.0  TRA

0075.0 +00.0000.0  ---
       -00.0000.0  ---

0076.0 +00.0000.0  ---
       -00.0000.0  ---

0077.0 +00.0000.0  ---
       -00.0000.0  ---
```

```
0100.0 +30.7772.0  FCA
       +57.0110.0  TRA

0101.0 -00.0000.0  ---
       -00.0000.0  ---

0102.0 +30.7766.0  FCA
       +57.0014.0  TRA

0103.0 +35.7766.0  FST
       +57.0114.0  TRA

0104.0 +04.7772.0  FAD
       +57.0130.0  TRA

0105.0 +04.0010.0  FAD
       +57.0043.0  TRA

0106.0 +40.0000.0  ---
       -00.0000.0  ---

0107.0 +00.0000.0  ---
       -00.0024.0  ---


0110.0 +07.7776.0  FMP
       +57.0176.0  TRA

0111.0 -00.0000.0  ---
       -00.0000.0  ---

0112.0 +00.7767.0  CLA
       +57.0154.0  TRA

0113.0 +43.0000.0  XAR
       +57.0131.0  TRA

0114.0 +30.7762.0  FCA
       +57.0125.0  TRA

0115.0 +43.0000.0  XAR
       +41.0043.0  ALS

0116.0 +01.7773.0  ADD
       +57.0134.0  TRA

0117.0 +00.7766.0  CLA
       +57.0135.0  TRA
```

```
0120.0 +35.7772.0  FST
       +57.0032.0  TRA

0121.0 -00.0000.0  ---
       -00.0000.0  ---

0122.0 +04.0226.0  FAD
       +57.0171.0  TRA

0123.0 +35.7762.0  FST
       +57.0244.0  TRA

0124.0 +41.0001.0  ALS
       +57.0240.0  TRA

0125.0 +05.7776.0  FDV
       +57.0017.0  TRA

0126.0 +60.0075.0  STO
       +57.0047.0  TRA

0127.0 -00.0000.0  ---
       -00.0000.0  ---


0130.0 +35.7772.0  FST
       +57.0140.0  TRA

0131.0 +41.0043.0  ALS
       +01.7773.0  ADD

0132.0 +40.0004.0  ARS
       +57.0053.0  TRA

0133.0 +07.0136.0  FMP
       +57.0122.0  TRA

0134.0 +60.7773.0  STO
       +57.0061.0  TRA

0135.0 +03.7777.0  SUB
       +57.0247.0  TRA

0136.0 +45.5136.0  ---
       -65.1770.0  ---

0137.0 +00.0000.0  ---
       -00.0000.1  ---
```

```
0140.0 +07.7776.0  FMP
       +57.0026.0  TRA

0141.0 -00.0000.0  ---
       -00.0000.0  ---

0142.0 +35.7774.0  FST
       +57.0152.0  TRA

0143.0 +35.0010.0  FST
       +57.0056.0  TRA

0144.0 +05.7776.0  FDV
       +57.0033.0  TRA

0145.0 +35.7760.0  FST
       +57.0156.0  TRA

0146.0 +60.0013.0  STO
       +57.0162.0  TRA

0147.0 +01.7770.0  ADD
       +57.7762.0  TRA


0150.0 +00.0000.0  ---
       -00.0000.0  ---

0151.0 -00.0000.0  ---
       -00.0000.0  ---

0152.0 +30.7760.0  FCA
       +57.0164.0  TRA

0153.0 +35.7776.0  FST
       +57.0165.0  TRA

0154.0 +66.0260.0  CTV
       +57.7771.0  TRA

0155.0 +33.0157.0  EXT
       +57.0273.0  TRA

0156.0 +51.0075.0  TMI
       +57.0167.0  TRA

0157.0 +00.0000.0  ---
       -77.7777.0  ---
```

```
0160.0 +60.0023.0  STO
       +57.0170.0  TRA

0161.0 +35.7763.0  FST
       +57.0173.0  TRA

0162.0 +00.7764.0  CLA
       +57.0145.0  TRA

0163.0 +04.0270.0  FAD
       +57.0175.0  TRA

0164.0 +35.7776.0  FST
       +57.0274.0  TRA

0165.0 +30.7764.0  FCA
       +57.0200.0  TRA

0166.0 +03.0070.0  SUB
       +57.0177.0  TRA

0167.0 +50.0075.0  TZE
       +57.0133.0  TRA


0170.0 +41.0024.0  ALS
       +57.0035.0  TRA

0171.0 +07.7760.0  FMP
       +57.0161.0  TRA

0172.0 +06.7774.0  FSB
       +57.0220.0  TRA

0173.0 +44.7763.0  FSQ
       +57.0163.0  TRA

0174.0 +34.0076.0  FCS
       +57.0104.0  TRA

0175.0 +04.0106.0  FAD
       +57.0272.0  TRA

0176.0 +06.7774.0  FSB
       +57.0024.0  TRA

0177.0 +51.0005.0  TMI
       +57.0007.0  TRA
```

```
0200.0 +05.7776.0  FDV
       +57.0067.0  TRA

0201.0 +00.0000.0  CLA
       +64.0300.0  CTL

0202.0 +57.0165.0  TRA
       +65.0300.0  CFL

0203.0 +60.7766.0  STO
       +57.0013.0  TRA

0204.0 +04.7772.0  FAD
       +57.0230.0  TRA

0205.0 +60.0150.0  STO
       +57.0117.0  TRA

0206.0 +66.0210.0  CTV
       +57.0124.0  TRA

0207.0 +00.7766.0  CLA
       +57.0224.0  TRA


0210.0 +00.0000.0  ---
       +65.0310.0  CFL

0211.0 +57.0100.0  TRA
       +77.0000.0  HTR

0212.0 +00.0000.0  ---
       -00.0005.0  ---

0213.0 +03.6000.0  ---
       -00.0000.0  ---

0214.0 +00.0000.0  ---
       -00.0000.0  ---

0215.0 -00.0000.0  ---
       -00.0000.0  ---

0216.0 +40.0000.0  ---
       -00.0000.0  ---

0217.0 +00.0000.0  ---
       -00.0001.0  ---

0220.0 +35.7762.0  FST
       +57.0030.0  TRA

0221.0 +30.7776.0  FCA
       +57.0241.0  TRA

0222.0 +64.0230.0  CTL
       +57.0147.0  TRA

0223.0 -00.0000.0  ---
       -00.0000.0  ---

0224.0 +40.0002.0  ARS
       +57.0246.0  TRA

0225.0 +01.0027.0  ADD
       +57.0036.0  TRA

0226.0 +40.6044.1  ---
       +35.1361.1  ---

0227.0 +00.0000.0  ---
       -00.0003.1  ---


0230.0 +35.7772.0  FST
       +57.7770.1  TRA

0231.0 +35.0076.0  FST
       +57.0245.0  TRA

0232.0 +60.7770.0  STO
       +00.7775.0  CLA

0233.0 +41.0023.0  ALS
       +60.7760.0  STO

0234.0 +01.7770.0  ADD
       +60.7770.0  STO

0235.0 +30.0201.0  FCA
       +01.7760.0  ADD

0236.0 +35.0251.0  FST
       +30.7776.0  FCA

0237.0 +35.0010.0  FST
       +57.0100.0  TRA
```

```
0240.0 +60.7775.0  STO
       +57.0255.0  TRA

0241.0 +05.7760.0  FDV
       +57.0231.0  TRA

0242.0 +35.7772.0  FST
       +57.0276.0  TRA

0243.0 +60.7767.0  STO
       +57.0155.0  TRA

0244.0 +00.7766.0  CLA
       +57.0257.0  TRA

0245.0 +07.7762.0  FMP
       +57.0242.0  TRA

0246.0 +45.0000.0  FNM
       +57.0123.0  TRA

0247.0 +40.0001.0  ARS
       +57.0060.0  TRA


0250.0 +00.0003.1  ---
       +77.7770.0  ---

0251.0 +00.0000.0  ---
       -00.0000.0  ---

0252.0 +00.0000.0  ---
       -00.0000.0  ---

0253.0 +00.7771.0  CLA
       +03.7770.0  SUB

0254.0 +51.0023.0  TMI
       +42.7772.1  STA

0255.0 +60.7771.0  STO
       +57.7771.1  TRA

0256.0 +01.0275.0  ADD
       +57.0205.0  TRA

0257.0 +03.7770.0  SUB
       +57.0206.0  TRA

0260.0 +00.0000.0  ---
       -00.0004.0  ---

0261.0 +40.0024.0  ARS
       +60.7763.0  STO

0262.0 +03.7766.0  SUB
       +51.0075.0  TMI

0263.0 +00.7770.0  CLA
       +01.7762.0  ADD

0264.0 +60.7767.0  STO
       +03.7766.0  SUB

0265.0 +51.0207.0  TMI
       +00.7766.0  CLA

0266.0 +01.7770.0  ADD
       +60.7766.0  STO

0267.0 +00.7763.0  CLA
       +57.7772.0  TRA


0270.0 +70.0000.0  ---
       -00.0000.0  ---

0271.0 +00.0000.0  ---
       -00.0001.1  ---

0272.0 +41.0002.0  ALS
       +57.0203.0  TRA

0273.0 +60.7762.0  STO
       +57.0112.0  TRA

0274.0 +34.0076.0  FCS
       +57.0204.0  TRA

0275.0 +00.0000.0  ---
       -00.0306.0  ---

0276.0 +00.7775.0  CLA
       +57.0222.0  TRA

0277.0 -00.0000.0  ---
       -00.0000.0  ---
```